Arab Republic of Egypt
Ministry of Education
Book Sector

MECHANICS

General Secondary Certificate
Third Form Secondary

2015 - 2016

دار مكة المكرمة للطباعة والنشر

Arab Republic of Egypt
Ministry of Education
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MECHANICS

GENERAL SECONDARY CERTIFICATE
(Third Form Secondary)

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2015 – 2016

المقـــدمة

يسعدنا أن نقدم لأبنائنا طلبة وطالبات الصف الثالث الثانوى هذا الكتاب فى الميكانيكا وفى ضوء توجيهات السيد الدكتور/وزير التربية والتعليم ، تمت إعادة تبويب الكتاب ومراجعته وتحديث مفردات المحتوى بما يجعل الكتاب فى صورة أدت إلى تبسيط عرض بعض الموضوعات وتزويد بعضها ببعض الأمثلة الإثرائية الحياتية.

وقد راعينا فى عرض موضوعاته التأكيد على المفاهيم والمهارات الرياضية التى تساعد الطالب على حل المشكلات معتمدًا فى ذلك على المستويات العليا للتفكير وليس الحفظ والاستظهار .

ولم نغفل فى عرض المادة العلمية ، ربط ما يدرسه الطالب بواقع حياته اليومية ، وقد راعينا أيضًا فى عرض الأمثلة والتمارين التدرج من السهل إلى الصعب ، ومراعاة الفروق الفردية بين الطلاب .

ويحتوى هذا الكتاب موضوعى الاستاتيكا والديناميكا :

أولاً : الاستاتيكا : عُرض هذا الجزء فى خمسة فصول، الفصل الأول: «الاحتكاك»، والفصل الثانى: «العزوم»، والفصل الثالث: «الاتزان العام»، والفصل الرابع: «القوى المتوازية المستوية»، والفصل الخامس: «الازدواجات».

ثانيًا : الديناميكا : عُرض هذا الجزء فى أربعة فصول، الفصل الأول: «قوانين نيوتن للحركة»، والفصل الثانى: «تطبيقات على قوانين نيوتن - الحركة على مستوى خشن»، الفصل الثالث: «الدفع والتصادم»، والفصل الرابع: «الشغل والقدرة والطاقة».

وقد ذيلنا هذا الكتاب بمجموعة من نماذج الاختبارات الخاصة بالاستاتيكا وكذلك الخاصة بالديناميكا والتى تؤكد على قياس نواتج التعلم المراد تمكين الطالب منها وهى مزودة بإجابات نهائية.

والله من وراء القصد

لجنة إعداد الكتاب

We are happy to present this book in mechanics to our sons and daughters, the students of the third form secondary, under the guidance of his excellency, the minister of education.

We rearranged the chapters of the book and added some life examples.

We concentrated on the basic concepts and the mathematical skills which help the student in solving problems depending on the high level skills.

This book contains two main branches:

First: Statics: which contains five chapters :

Chapter one : Friction

Chapter two : Moments

Chapter three : Parallel coplanar forces

Chapter four : General Equilibrium

Chapter five : Couples

Second: Dynamics : which contains four chapters:

Chapter one : Newton's laws of motion

Chapter two : Applications on Newton's lows – motion on a rough plane

Chapter three : Impulse and collision

Chapter four : Work – Power – Energy

At the end of the book you will find some exam models and final answers.

Committee of preparing
The book

Contents

Part 1

Statics

Chapter One
Friction

Preface :

In this chapter we will deal with friction force and its properties.

Objectives:

By the end of teaching this chapter, the student should be able to:

(1) recognize the action force and its direction and the reaction force and its direction between two bodies in state of touch in the two cases when [the two bodies are smooth or rough] .

(2) recognize properties of friction and limiting friction.

(3) recognize coefficient of friction and angle of friction.

(4) recognize conditions of equilibrium of a body on a rough inclined plane.

(5) recognize the relation between the angle of friction and the angle of inclination of a plane to the horizontal when putting a body on a rough inclined plane.

Topics :

(1) Properties of friction force.

(2) Coefficient of friction.

(3) Angle of friction.

(4) Equilibrium of a body on a rough horizontal plane.

(5) Equilibrium of a body on a rough inclined plane.

Friction

Here we consider the phenomenon of friction and its properties. We Shall assume that all the bodies we are dealing with behave like a particle, that is like a mass concentrated at a geometrical point.

Reaction :

Newton's third law of motion is often stated as " For every action there is a reaction equal in magnitude and opposite in direction ". That is, when a body acts with a force (action) on another body, the latter, in turn, acts on the first body with a force (reaction) which is equal to the first in magnitude but opposite in direction.

For example, for a ball resting on a table, Fig. (1), the ball is acting with a force (action) on the table, while the table is acting with a force (reaction) on the ball whose magnitude is represented by R . These two forces are equal in magnitude but opposite in direction.

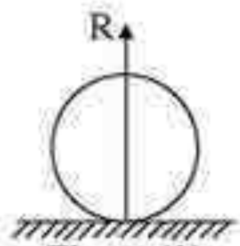


Figure (1)

It is important to notice that these two forces do not act on the same body. One of the force (action) is acting on the table while the other one (reaction) is acting on the ball.

The Force of Friction :

Consider a cube (made of wood or glass) resting on a horizontal table. If the cube is give a push, the cube will come to rest after sliding over the table for a certain distance, during which its velocity decreases to zero. This indicates the existence of a force which resists the motion of the cube on the table and brings it to a stop. This force is called (the force of friction between the cube and the table). Friction, in general, plays an important role in our practical life. If there was no friction, we would not be able to walk on the ground without sliding, or a railway engine would be unable to pull a train without the wheels sliding, ... ect . . .

Smooth and Rough Bodies :

The action and reaction between any two bodies, in contact with each other, depends upon the nature of the two bodies and upon the other forces acting on them.

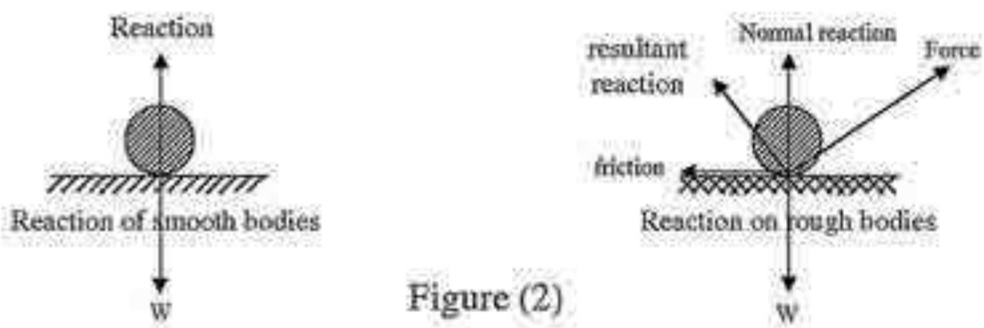


Figure (2)

For smooth bodies the reaction is normal to the common tangent plane to the surfaces of the two bodies in contact. On the other hand, when we try to move rough bodies, the reaction force would have a component parallel to the common tangent plane, which is the force of friction, as well as a component normal to the common tangent plane, which is the normal reaction. In the latter case the reaction force is known as the resultant reaction, which is replaced in many cases by its resolved components viz: the force of friction and the normal reaction.

Fig. (2) shows the reaction force of a table on a body resting on it.

Experiment :

Lay a block of wood on a horizontal table, attach the block to a string passing over a smooth pulley at the edge of the table, and attach a scale pan to the vertical loose end of the string, as in Fig. (3).

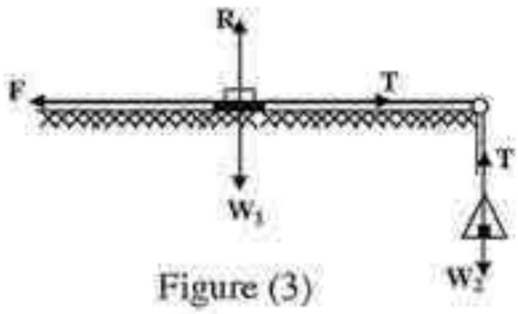


Figure (3)

Place a suitable weight on the block, then gently put a small weight in the scale pan. We notice that the block does not move, which implies that, despite the existence of the tension T in the string, the frictional force acting on the block was big enough to hold the block in place.

As we know, this tension is equal to the net weight of the scale pan and the weights it contains.

By increasing the weights in the scale pan gradually we notice that, at a certain value the block will start to move over the table. This implies that the value of the force of friction increases gradually with the applied tension, until it reaches a maximum or a limiting value which it does not exceed.

It is when the tension exceeds this limit that the force of friction cannot balance it so the block begins to move.

It is found that by increasing the weights placed on the block we have to increase the weights in the pan so that the motion of the block is about to move.

Fig. (3) shows that forces acting on the block are :

(1) The net weight of the block and the weights placed on it "W_1".

(2) The tension in the string "T", which is equal to the weight of the pan and the added weights.

(3) The force of friction F, which is parallel to the table surface i . e . horizontal.

(4) The normal reaction R.

At equilibrium, the direction of the frictional force is opposite to that of the tension, and the normal reaction is opposite to the weight of the block and we have the two equalities :

$$R = W_1 \qquad , \qquad F = T$$

From the previous experiment we have the following conclusions for the properties of friction.

properties of friction :

(1) The force of friction always acts in the direction opposite to the direction the body is tending to move.

(2) The force of friction increases with the tangential force that tends to move the body so that the two force are equal provided that the body is in a state of equilibrium.

(3) The magnitude of the force of friction increases up to a certain limit which it does not exceed. At this value, the motion is just about to begin and the friction is then called the limiting friction. When motion takes place the magnitude of the force of friction is nearly equal to its maximum value in other words during motion the friction is a limiting friction.

(4) The magnitude of the limiting friction bears a constant ratio to the normal reaction. This ratio depends on the nature of the two surfaces in contact and is independent of their shapes and masses.

Coefficient of Friction :

The ratio between the magnitude of the limiting friction and the magnitude of the normal reaction is called " the coefficient of friction ".

Hence if F is the magnitude of the limiting friction, R is the magnitude of the normal reaction and μ is the coefficient of friction, we have :

$$\boxed{\mu = \frac{F}{R}}$$

i . e . $F = \mu R$

Which is the maximum amount attained by the force of friction.

It is important to bear in mind that this equality holds only at the limiting friction i . e . it is only valid when motion is about to start or when motion is set on by action. Therefore in general we may write the inequality :

$$\boxed{F \leq \mu R}$$

Angle of Friction :

Let R' be the magnitude of the resultant reaction and θ be the angle that it makes with the normal reaction as shown in Fig. (4).

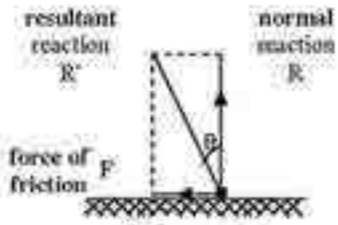


Figure (4)

As the magnitude of the force of friction increases , the value of θ increases (the normal reaction is assumed constant) until it reaches a maximum value when the magnitude of the force of friction assumes its maximum value μ R, i.e. when the friction is limiting. It is then that the angle between the resultant reaction and the normal reaction is called the angle of friction', and its measure is represented by λ , see Fig. (5) then we have :

$$\tan \lambda \;=\; \frac{F_L}{R} = \mu$$

i . e . " the tangent of the angle of friction equals the coefficient of friction ".

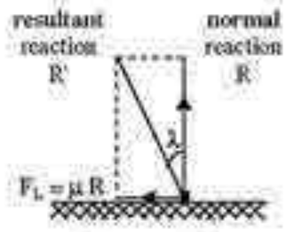


Figure (5)

Resultant reaction force:

The resultant reaction force $R^{\backslash}$ is the resultant of the friction and the normal reaction force. Its magnitude is given by

$$R^{\backslash} = \sqrt{R^2 + F^2}$$

When the friction force is limiting we have

$$R^{\backslash} = \sqrt{R^2 + \mu^2 R^2} = R\sqrt{1 + \mu^2}$$

Equilibrium of a particle on a horizontal rough plane :

Suppose a particle of weight W is in equilibrium on a horizontal rough plane and acted upon by a force $\overrightarrow{P}$ inclined by an angle θ with the horizontal Fig. (6).

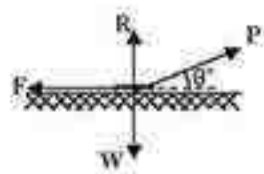


Figure (6)

Now the particle is in equilibrium under the following forces.

(1) the weight $\overrightarrow{W}$ which is directed vertically downwards.

(2) The given force $\overrightarrow{P}$.

(3) The resultant reaction due to the effect of the plane on the particle, and this may be resolved into two components, the normal reaction R which is directed vertically upwards and the force of friction F which is opposite to the direction in which the particle tends to move.

Since the particle tends to move on the plane under the action of the tangential component of P (of magnitude $P \cos \theta$), therefore the force of friction is directed opposite to this component as shown in Fig. (7).

By resolving the forces action on the particle into components along and perpendicular to the plane, the equations for equilibrium are :

$$F = P \cos \theta$$

$$R + P \sin \theta = W$$

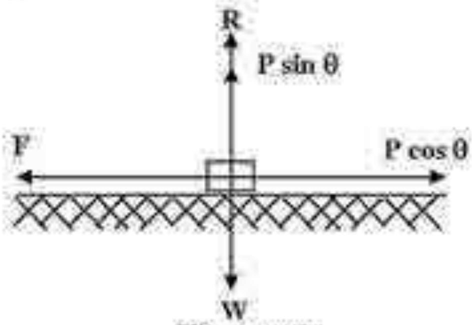


Figure (7)

These two relations together determine the magnitudes of the normal reaction and the force of friction.

- The particle will keep contact with the plane as long as :

$$R > 0 ,$$

i.e $\quad W > P \sin \theta$

- If the motion is just about to take place under the action of the tangential component of the force $\overrightarrow{P}$, then the friction is limiting and the above relations are still true with $\quad F = \mu R$
- If the applied force $\overrightarrow{P}$ is horizontal, then we put $\theta = 0°$ in the above relations.

Example (1) :

A wooden block of weight 6 kg.wt. is placed on a horizontal table, and is connected by a string passing over a smooth pulley at the edge, to a weight of magnitude 1.5 kg.wt. which is hanging freely. Given that the block is in equilibrium, find the force of friction and the normal reaction.

If the coefficient of friction between the block and the table is $\frac{1}{3}$, state whether or not the body is about to move.

Solution :

The force that tends to move the block on the table is the pull of the horizontal string whose magnitude is 1.5kg. w.t., so that the force of friction F acts in the opposite direction as shown in Fig. (8).

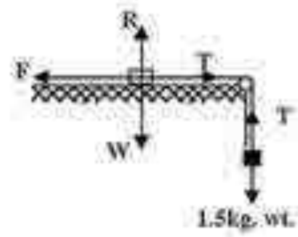


Figure (8)

Putting T = 1.5 kg.wt, and $\theta = 0°$ in the previously obtained relations we have :

$$F = 1.5 \cos 0° = 1.5 \text{ kg.wt}$$

$$R + 1.5 \sin 0° = 6 \text{ kg.wt.}$$

$$\therefore R = 6 \text{ kg.wt}$$

To find out whether or not the motion is about to start, we calculate the limiting friction which is μR .

$$\therefore \quad \mu R = \frac{1}{3} \times 6 = 2 \text{ kg.wt}$$

$$\therefore \quad F < \mu R$$

Therefore the friction is not limiting and the body is not about to move.

Equilibrium of a body on a rough inclined plane :

Consider a body in equilibrium on a rough plane inclined to the horizontal with an angle its measure θ. The body is in equilibrium on the plane under the action of two forces.

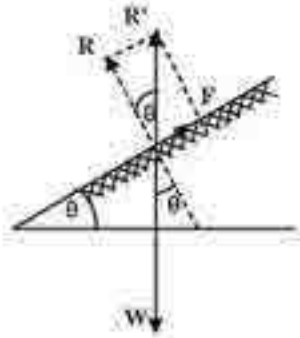


Figure (9)

(1) Its weight $\vec{W}$ which is acting vertically downwards, and its magnitude W

(2) The resultant reaction R'.

It follows from the condition of equilibrium that the resultant reaction is acting vertically upwards and ;

$$R' = W \qquad (1)$$

Now the force of friction and the normal reaction may easily be found by considering them as the components of the resultant reaction along and normal to the inclined plans as shown in Fig. (9).

For the force of friction we have :

$$F + R' \sin\theta = W \sin\theta \qquad (2)$$

This force is in a direction opposite to the direction of the possible motion of the body i . e . is parallel to the line of greatest slope of the plane and is directed upwards.

For the normal reaction we have :

$$R + R' \cos\theta = W \cos\theta \qquad (3)$$

In case the body is about to slide down we have the following rule :

Rule :

If a body placed on a rough inclined plane and is about to slide down under the action of its weight only then the measure of the angle of inclination of the plane is equal to the angle of friction.

Proof :

Since the friction is limiting then the resultant friction makes with the normal to the plane an angle equal to the angle of the friction. Referring to Fig. (10), we find that ;

$$\theta = \lambda$$

Where θ is the angle of inclination of the plane to the horizontal.

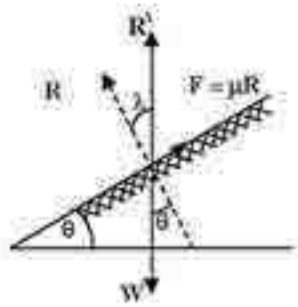


Figure (10)

This result may be expressed in terms of the coefficient of friction as :

$$\boxed{\tan\theta = \mu} \Rightarrow \boxed{\mu = \tan\lambda}$$

Experiment (1) :

Determination of the coefficient of friction between two contacting surfaces, using the inclined plane.

To determine the coefficient of friction between two surfaces in contact, using the inclined plane.

Equipment required for the experiment

A rough plane – a wooden block with one face plain on the opposite face has a rectangular hole – chabistan clamp with a pivot – a protractor – a plumb line .

Method :

1. Clamp the pivot to the stand and affix the plane to it.
2. Affix the protractor to the plane so that its straight edge is. levelled with the edge of the plane as shown in Fig. (11).
3. Hang the plumb line from the center of the protractor so that the line is in alignment with the midline of the protractor when the plane is horizontal.

Figure (11)

4. Set the plane in a horizontal position and place the block with its plane face resting on it, then put a suitable weight in the hollow on the opposite face of the block.
5. Gradually tilt the plane until the block begins to slip down with help of slight tapping.
6. Find the angle of inclination of the plane to the horizon indicated by the plumb line on the protractor.
7. Repeat the experiment for different weights in the hollow and each time record the angle made by the plane to the horizon.

   it will he noticed that all the recorded angles are nearly the same.
8. The angle of friction is the average value of these measurements.
9. the coefficient of friction is found by calculating the tangent of the angle of friction.

Example (2) :

A wooden block of weight 2 kg.wt rests in equilibrium on a plane, inclined at 30° to the horizontal, under the action of a force whose magnitude is 2.5 kg.wt and whose direction is that of the line of greatest slope upwards. If the coefficient of friction is 0.9, find the force of friction. State whether or not the motion is about to begin.

Solution :

The tangential forces that tend to move the block on the plane are :

(1) The tangential component of the weight which is directed downwards along the line of greatest slope, and of magnitude given by :

$$W \sin 30^\circ = 2 \times \frac{1}{2} = 1 \text{ kg.wt}$$

(2) and the given applied force acting along the line of greatest slope upwards and whose magnitude is 2.5 kg.wt

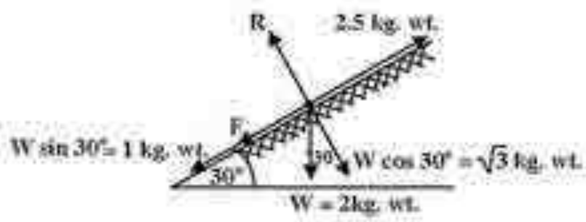


Figure (12)

The block tends to move up the plane and along the line of greatest slope because the second force is greater than the first one. It follows that the force of friction must act downwards along the same line as shown in Fig. (11).

For equilibrium

$$2.5 = F + 1 \qquad F = 1.5 \text{ kg.wt}$$

also

$$R = W \cos 30^\circ = 2 \times \frac{\sqrt{3}}{2} = \sqrt{3} \text{ kg.wt} \simeq 1.732 \text{ kg.wt}$$

To find out whether or not the force of friction is limiting, we must calculate its maximum possible value i . e.

$\mu\ R = 0.9 \times 1.732 = 1.559$ kg. wt.

$$F < \mu\ R$$

It follows that the friction is not limiting and hence the motion is not about to begin.

Example (3) :

A body of weight 76 Newtons is about to move under its own weight when placed on a rough plane which is inclined to the horizontal with an angle whose tangent is $\frac{1}{4}$. if the body is placed on a horizontal plane which is as rough as the inclined plane and is acted on by an upward pull in a direction inclined to the horizontal with an angle whose sine is $\frac{3}{5}$ so that the body is about to move.

Find the magnitude of this force and the normal reaction.

Solution :

The coefficient of friction = the tangent of the angle of inclination of the plane to the horizontal = $\frac{1}{4}$ .Since the body is about to begin, hence the friction is limiting and is equal to $\frac{1}{4}$ R.

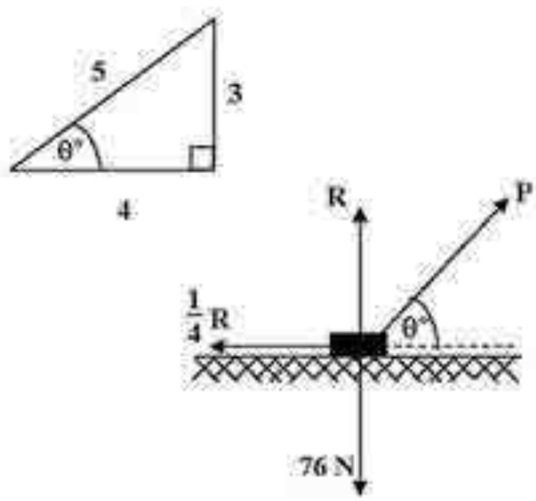


Figure (13)

let P be the magnitude of the pull whose direction is inclined to the horizontal at an angle θ fig.(13). By resolving in the horizontal and vertical directions as fig.(14) we get :

$$P \cos \theta = \frac{1}{4} R$$

i.e $P \times \frac{4}{5} = \frac{1}{4} R \quad \therefore R = \frac{16}{5} P \quad (1)$

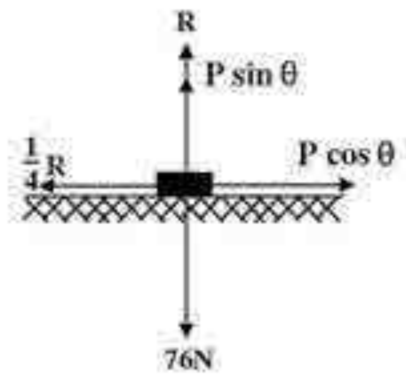


Figure (14)

On the other hand :

$$R + P \sin \theta = 76$$

$$\because R + P \times \frac{3}{5} = 76$$

$$\therefore R = 76 - \frac{3}{5} P \qquad (2)$$

Substitution from (1) in (2) given

$$\frac{16}{5} P = 76 - \frac{3}{5} P$$

$$\therefore \frac{19}{5} P = 76 \qquad \therefore P = 20 \text{ Newtons}$$

$$\therefore R = \frac{16}{5} \times 20 = 64 \text{ Newtons}$$

Example (4) :

A body of weight 40 Newtons rests on a rough plane which is inclined to the horizontal at an angle 30°. The body is pulled up by a string which makes an angle 30° with the plane. The string lies in the vertical plane which contains the body and the line of greatest slope. If the coefficient of friction is $\frac{1}{4}$.

Prove that the least pull in the string which prevents the load from moving down the incline is about 15.3 Newtons.

Solution :

Since the load is about to slide down, the friction is limiting and acts upwards, so the applied forces are :

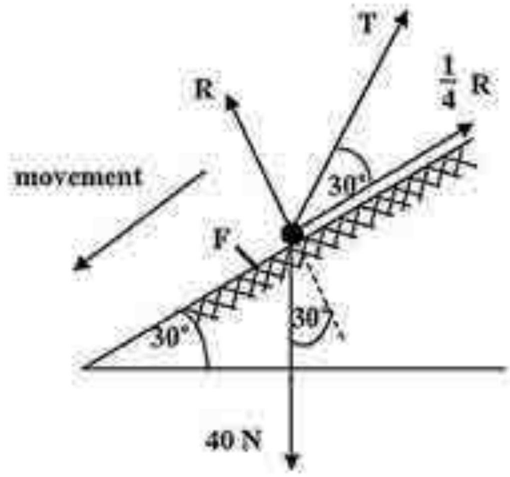


Figure (15)

(1) The weight 40 Newtons, vertically downwards.

(2) The normal reaction R, perpendicular to the plane as in Fig. (15).

(3) The limiting friction $\frac{1}{4}$ R, which is directed upwards.

(4) The tension T in the string inclined at 30° to the plane and this is the least tension which prevents sliding.

Resolving along the plane and the perpendicular to the plane, we write down the equilibrium equations i . e. :

$$\frac{1}{4} R + T \cos 30^\circ - 40 \sin 30^\circ = 0 \qquad (1)$$

$$R + T \sin 30^\circ - 40 \cos 30^\circ = 0 \qquad (2)$$

Multiplying (1) by 4, then subtracting (2) from the resulting equation, we get:

$$4 T \cos 30^\circ - T \sin 30^\circ = 160 \sin 30^\circ - 40 \cos 30^\circ$$

$$T \left(4 \times \frac{\sqrt{3}}{2} - \frac{1}{2} \right) = 160 \times \frac{1}{2} - 40 \times \frac{\sqrt{3}}{2}$$

$$\therefore \ T = \frac{80 - 20\sqrt{3}}{2\sqrt{3} - \frac{1}{2}} = \frac{45.36}{2.964} \simeq 15.3 \text{ Newtons}$$

Example (5) :

A body of weight 50 Newton is placed on a rough inclined plane and is acted on by a force P along the line of greatest slope upwards . Given that the body is about to move upwards when P = 30 Newtons, and it is about to move downwards when P = 20 Newtons, Find the coefficient of friction between the body and the plane.

Solution :

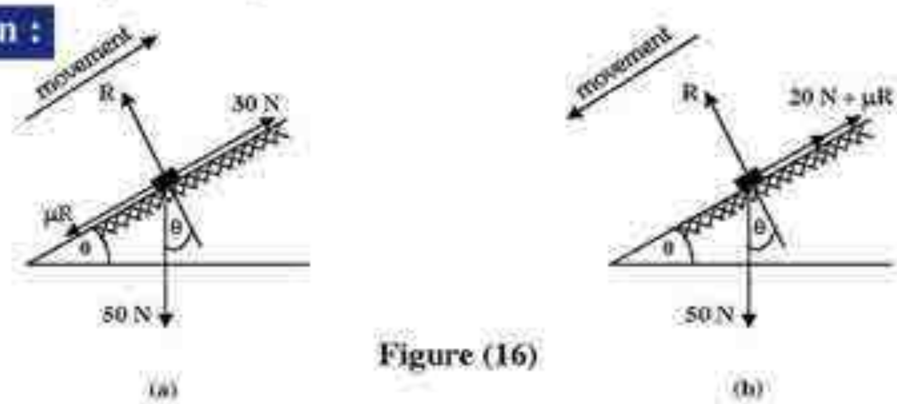


Figure (16)

In Fig. (14). (First) P = 30 Newtons. The motion is about to begin upwards, the friction is limiting and acts downwards.

Resolving along the plane and the perpendicular to the plane, the equilibrium equations are :

$R = 50 \cos \theta$ (1)

$30 = \mu R + 50 \sin \theta$ (2)

where θ is the angle of inclination of the plane. to the horizontal and in fig. (16a). (Second) P = 20 Newtons . The motion is about to begin downwards, the friction is limiting and acts upwards.

Resolving along the plane and the perpendicular to the plane, the equation for equilibrium are :

$R = 50 \cos \theta$ (3)

$20 + \mu R = 50 \sin \theta$ (4)

The sum of (2) and (4) gives :

$$50 = 100 \sin \theta \qquad \therefore \sin \theta = \frac{1}{2} \qquad \therefore \theta = 30^\circ$$

Substituting in (1) we get :

$$R = 50 \cos 30^\circ = 50 \times \frac{\sqrt{3}}{2} = 25\sqrt{3} \text{ Newtons}$$

Substituting in (2) we get :

$$30 = \mu \times 25\sqrt{3} + 50 \times \frac{1}{2}$$

$$\therefore 5 = 25\mu\sqrt{3} \Rightarrow \mu = \frac{1}{5\sqrt{3}} \qquad \therefore \mu = \frac{\sqrt{3}}{15}$$

Example (6) :

When a weight W is placed on a rough plane inclined at an angle θ to the horizontal, it is found that the weight is about to slide down. Prove that the least force along the line of greatest slope which makes the weight about to move upwards is equal to 2 W sin θ. Prove also that the resultant reaction is equal to W.

Solution :

Let P represents the required force. Now $\mu = \tan \theta$ (1) by resolving along the plane and the perpendicular to the plane we have, for equilibrium, the two conditions.

$$R = W \cos \theta, \qquad (2)$$

$$P = \mu R + W \sin \theta \qquad (3)$$

Substituting from (1), (2), (3) we get

$$P = \tan \theta \times W \cos \theta + W \sin \theta$$

$$= \frac{\sin \theta}{\cos \theta} \times W \cos \theta + W \sin \theta$$

$$= W \sin \theta + W \sin \theta = 2 W \sin \theta$$

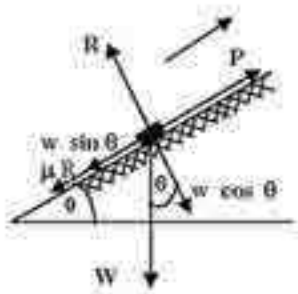


Figure (17)

To find the resultant reaction, we notice that the angle of friction is equal to the angle of inclination of the plane to the horizon i . e. $\lambda = \theta$. So that the resultant reaction R' is give by : $\because \frac{R}{R'} = \cos \lambda$

$$R' = \frac{R}{\cos \lambda} = \frac{R}{\cos \theta} = \frac{W \cos \theta}{\cos \theta} = W$$

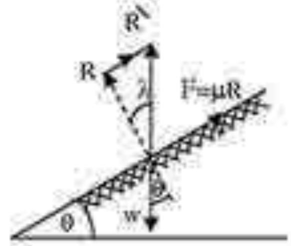


Figure (18)

Example (7) :

A body of weight 6 Newton rests on a rough horizontal plane. Two forces 2 and 4 Newton act horizontally on the body, the angle between them being 120°. If the body is in equilibrium, prove that the angle of friction λ is not less than 30°.If λ=45°and the two forces act in their previous direction and the 4 Newton. force remains constant, find the least value for the other force to move the body and determine the direction of motion.

Solution :

As the body is not in limiting equilibrium the force acting on it are its weight W vertically , down the reaction R perpendicular to the plan. The two force 2 and 4 Newton, and the force of friction F acting on the horizontal plane perpendicular to R. Fig. (19).

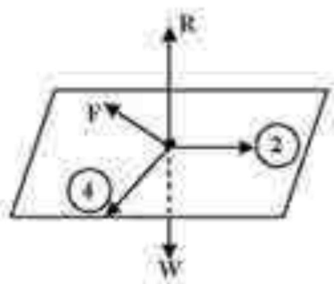


Figure (19)

The force perpendicular to the plane cancel each other

$\therefore R = W$ $\qquad\qquad \therefore R = 6$ Newtons

The three force 2, 4 and F act in one plane and are in equilibrium

$$\therefore F = \sqrt{F_1^2 + F_2^2 + 2 F_1 F_2 \cos \theta}$$

$$= \sqrt{(2)^2 + (4)^2 + 2 \times 2 \times 4 \cos 120^\circ}$$

$$= \sqrt{4 + 16 - 8} = 2\sqrt{3} \text{ Newton}$$

But $\mu R \geqslant F$ as F is not limiting friction

$\therefore \mu \times 6 \geqslant 2\sqrt{3}$ $\qquad \therefore \mu \geqslant \frac{\sqrt{3}}{3}$

$\therefore \tan \lambda \geqslant \tan 30^\circ$ $\qquad \therefore \lambda$ is not less than 30°

When the body is about to move :

$\therefore \mu = \tan 45^\circ = 1$, $F = \mu R = 1 \times 6 = 6$ Newtons $\quad F_1 = 4$ Newtons

$\therefore F = \mu R = 6$

$\therefore 6^2 = 4^2 + F_2^2 + 2 \times 4 \times F_2 \cos 120^\circ$

$\therefore F_2^2 - 4 F_2 - 20 = 0$

$\therefore F_2 = \frac{4 \pm \sqrt{16 + 80}}{2} = \frac{4 \pm \sqrt{96}}{2}$ $\qquad \therefore F_2 = 2(1 + \sqrt{6})$ Newton

As the direction is opposite to the expected direction of motion, then the direction where the body is about to move is opposite to F.

i . e. in the direction of the resultant of the two forces

$F_1 = 4$, $F_2 = 2(1 + \sqrt{6})$ i . e. in direction inclined at

F_1, by angle Φ where :

$$\tan \Phi = \frac{F_2 \sin 120^\circ}{F_1 + F_2 \cos 120^\circ}$$

$$= \frac{2(1 + \sqrt{6}) \frac{\sqrt{3}}{2}}{4 + 2(1 + \sqrt{6}) \times -\frac{1}{2}} = \frac{(1 + \sqrt{6})\sqrt{3}}{3 - \sqrt{6}} = \frac{(1 + \sqrt{6})\sqrt{3}}{\sqrt{3}(\sqrt{3} - \sqrt{2})} = \frac{1 + \sqrt{6}}{\sqrt{3} - \sqrt{2}}$$

$\therefore \Phi \simeq 84^\circ \; 44'$

Exercises (1)

(1) A body of weight 240 kg.w is placed on a horizontal rough plane it is required to pull it by a string which makes an angle of $30°$ with the horizontal. If the coefficient is 0.3, calculate correct to two decimal places, the least tension that makes the body about to move.

(2) A body of weight 3 Newtons is placed on a plane inclined at $30°$ to the horizontal and the coefficient of friction between the weight and the plan is $\frac{2}{3}$. A 2 Newtons force is acting on the body upwards along the line of greatest slope. Given that the body is in equilibrium, find the force of friction and investigate whether or not the body is about to move.

(3) A body of weight 4 Newtons is placed on a plane inclined at $30°$ to the horizontal and the coefficient of friction between the plane and the body is $\frac{\sqrt{3}}{4}$. A $\frac{1}{2}$ Newtons force is acting on the body upwards along the line of greatest slope. Given that the body is in equilibrium, find the force of friction. Is the body about to move?

(4) A body of weight 38 Newtons is about to move under its own weight when placed on a rough plane inclined to the horizontal at an angle whose tangent is $\frac{1}{5}$. If this body is placed on a horizontal plane which is as rough as the inclined plane, and is acted on by a force inclined to the horizontal at an angle whose sine is $\frac{4}{5}$ so that the body is about to move. Find the force and the normal reaction.

(5) A body of weight 190 Newtons rests on a plane whose inclination to the horizontal has a tangent equals $\frac{5}{12}$, and the coefficient of friction between the body and the plane is $\frac{1}{2}$. Find the least horizontal force, in the vertical plane through the line of greatest slope, that will make the body is about to move upwards.

(6) A body of weight 30 Newtons is placed on a rough inclined plane. When the plane is inclined at 30° to the horizontal, the body is about to slide down. If the inclination of the plane to the horizontal is increased to 60° calculate the least force acting on the body parallel to the line of greatest slope, such that

(a) The body is about slide down.

(b) The body about to move upwards.

(7) A 2 kg. weight is placed on a horizontal rough plane. The plane is tilted gradually, so the weight is about to slide down when the inclination of the plane is 30° to the horizontal. Prove that the coefficient of friction is equal to $\frac{\sqrt{3}}{3}$. If the weight is then attached to a string which is pulled in a direction inclined at 60° to the horizontal so that the body is about to move upwards. Given that the string is in the vertical plane through the line of greatest slope, calculate the tension in the string and prove that the force of friction is $\frac{1}{2}$ kg. wt.

(8) A body whose weight is 20 Newton is placed on a rough plane inclined to the horizontal by an angle whose tangent is $\frac{4}{3}$. If F_1 is the least force if applied along the line of greatest slope upwards, the body is about to move downwards. F_2 is the least force if applied horizontally, the body is about to move downwards. If $F_1 = F_2$ then find the coefficient of frication of the rough plane and the magnitude of any of the two forces.

(9) A 130 Newtons weight is placed on a rough plane which is inclined to the horizontal by an angle whose cosine is $\frac{5}{13}$. A force is applied to the weight parallel to the line of greatest slope upwards. If the coefficient of friction between the weight and the plane is $\frac{2}{5}$, then find the limits between which the applied force lies, so as to make the weight about to move.

(10) A body whose weight is W is placed on a rough plane inclined at angle θ to the horizontal and the angle of friction is λ. A force $\overrightarrow{F}$ acts on the body parallel to the plane to prevent the body from slipping.

Prove that $F = \frac{W \sin(\theta - \lambda)}{\cos \lambda}$

(11) A body whose weight is 4 kg. wt. is placed on a rough plane inclined at an angle 30° to the horizontal, and the coefficient of friction between the body and the plane is $\frac{\sqrt{3}}{2}$. Find whether the body is slipping or just about to slip on the plane or the friction is not a limiting. Hence. Find the magnitude and the direction of the force of friction then.

If a force acts parallel to the line of greatest slope of the plane, find the magnitude and direction of this force :

a. to make the body about to move up the plane.

b. to make the body move down the plane.

(12) A body whose weight is W is placed on a horizontal rough plane. Two perpendicular forces F_1 & F_2 act horizontally on the body and keep it in equilibrium. Prove that the coefficient of friction μ between the body and the plane is at more than $\frac{\sqrt{F_1^2 + F_2^2}}{W}$

If $F_2 = \frac{W}{2}$, $\mu = 1$ and F_1 is increased gradually, till the body is about to move, then find the magnitude of F_1 and the direction in which the body is about to move.

Chapter two

Moment

Preface :

In this chapter we will deal with the concept of the moment of a force about a point .

Objectives:

By the end of teaching this chapter, the student should be able to:

(1) Find the moment of a force with respect to a point.

(2) Determine the angle between $\overrightarrow{r}$ and $\overrightarrow{f}$ when calculating the moment of the force with respect to a point.

(3) Determine the magnitude and the direction of the moment of a force with respect to a point.

(4) Calculate the moment of coplanar forces about a point in their plane.

(5) Find the sum of moments of a system of forces about any point in their plane.

(6) Solve life problems in moment.

Topics :

(1) Scalar product of two vectors.

(2) Vector product of two vectors.

(3) The moment of a force about to a point.

(4) Moments of coplanar forces.

Moments

The effects of forces on rigid bodies is not the same as its effect on a particle. When a force acts on a particle, it moves in the direction of the force (unless the particle is under the effect of other forces). But when a force acts on a rigid body (which occupies a finite non-zero size of space) the resulting motion may be so simple or so complicated, depending on the shape of the body and on the point of application of the force on the rigid body.

The student can experimentally recognize different possible motions of a rigid body under the action of a given force.

As an example if we place a uniform rod $\overline{AB}$ on a smooth horizontal table, and if we act on the rod by a force at its centre C in a direction perpendicular to AB in the plane of the table as in Fig. (20 a) we notice that the rod moves on the table so that it is always parallel to $\overleftrightarrow{AB}$. Such a motion is called "Translation motion". If the end B is attached

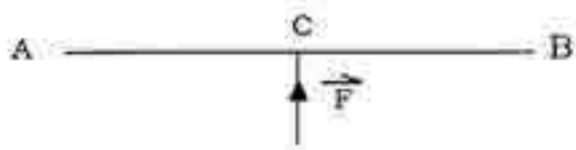


Fig. (20 a)

to a hinge fixed in the table so that the rod can rotate freely about B, we notice that the rod starts to rotate about B under the effect of the force $\vec{F}$. Such a motion is called "Rotational motion". If we now let the end B free and place the rod on the table and the force $\vec{F}$ acts as in Fig. (20-b), we notice that the rod acquires a motion which is a mixture of translation and rotation.

Fig. (20 b)

Now let us examine carefully the capability of a force for producing a rotational motion. If we consider once more the rod AB placed on the table with its end B fixed to the table by a hinge so that it can rotate about B in the plane of the table. Using the hinge at B cancels the translational motion, and we are left with the rotational motion about B. We notice now that when the point of application of the force is at A, the force is more capable of rotating the rod, and that this capability diminishes when the point of application of the force is nearer to B, and finally when the force acts at B, the rod does not move at all. We notice also that when we fix the point of application and increase the magnitude of the force, its capability for rotating the rod increases.

It seems now that there are two factors governing the capability of a force for producing rotation of the rod about its end B.

1. The magnitude of the force.
2. The distance between its point of application from the hinge.

We shall define a vector quantity called MOMENT OF A FORCE WITH RESPECT TO A POINT, which defines the capability of a force in producing a rotation to the body on which it acts.

But before we gave the precise definition to the moment of a force we have to complete our knowledge about vector algebra by defining two kinds of vector multiplication, which are :

Scalar product of two vectors

Vector product of two vectors.

The scalar product of two vectors has been discussed in the course of analytical geometry in the first form.

I Scalar product of two vectors :

Let $\vec{a}$ and $\vec{b}$ be two non-zero vectors, also let $a = \|\vec{a}\|$, $b = \|\vec{b}\|$ and θ the smaller angle between these two vectors drawn outwords from the same point as in figure (21).

Definition :

The scalar product of the two vectors $\vec{a}$ and $\vec{b}$ denoted by $\vec{a} \cdot \vec{b}$

$$\boxed{\vec{a} \odot \vec{b} \quad = \quad a\, b \cos\theta}$$

N.B.1: $\vec{a} \odot \vec{b} = \vec{b} \odot \vec{a}$

N.B.2: $\vec{a} \odot \vec{o} = \vec{o} \odot \vec{b} = \vec{o} \odot \vec{o} = o$

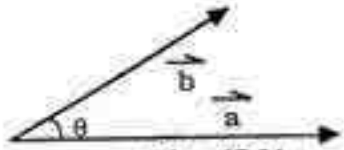


Fig. (21)

The algebraic projection (algebraic component) of a vector on the direction of another vector :

Definition :

The algebraic projection (algebraic component) of vector $\vec{b}$ in the direction of vector $\vec{a}$, (sometimes we say the algebraic projection of $\vec{b}$ on $\vec{a}$) is defined to be the scalar quantity ($b \cos\theta$). We notice that the algebraic projection is positive when θ is an acute angle, (Fig. 22a), negative is an obtuse angle (Fig. 22 b), and if θ is a right angle, then the algebraic projection is equal to zero.

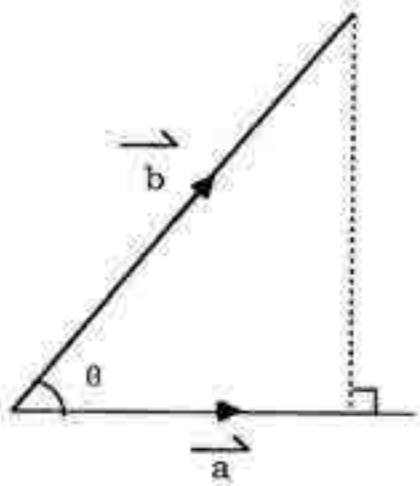


Fig. (22 a)

Noticing that

$$\vec{a} \odot \vec{b} = a\,(b \cos \theta) = b\,(a \cos \theta)$$

i.e. the scalar product of two non-zero vectors is equal to the product of the magnitude of one of them times the algebraic projection of the other in the direction of the first.

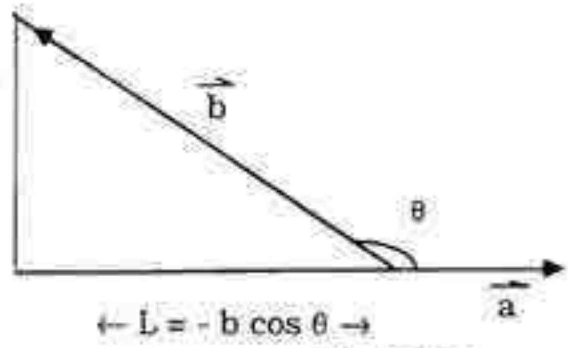


Fig. (22 b)

Theorem 1 :

For any vector $\vec{a}$

$$\vec{a} \odot \vec{a} = a^2$$

Result :

If $\hat{i}$, $\hat{j}$ are two unit vectors in the two perpendicular direction $\overrightarrow{OX}$, $\overrightarrow{OY}$ respectively then

$$\hat{i} \odot \hat{i} = \hat{j} \odot \hat{j} = 1 \quad \text{and} \quad \hat{i} \odot \hat{j} = 0$$

Theorem 2 : (multiplication by a constant)

For two vectors $\vec{a}$, $\vec{b}$

$$(k\,\vec{a}) \odot \vec{b} = \vec{a} \odot (k\,\vec{b}) = k\,(\vec{a} \odot \vec{b}) \text{ where } k \neq 0.$$

Theorem 3 :

For any three vectors $\vec{a}$, $\vec{b}$, $\vec{c}$

$$(\vec{a} + \vec{b}) \odot \vec{c} = \vec{a} \odot \vec{c} + \vec{b} \odot \vec{c} \qquad \text{(Distributivity)}$$

Remark : If $\vec{a} = (x_1 , y_1)$ and $\vec{b} = (x_2 , y_2)$, then

$$\vec{a} \odot \vec{b} = x_1 x_2 + y_1 y_2$$

Resolution of a vector into two perpendicular directions using scalar product of two vectors :

We can resolve a vector in two perpendicular directions using scalar product of two vectors.

As an example consider a force represented by the vector $\vec{F}$, and we wish to resolve this vector in the two perpendicular directions $\overrightarrow{OX}$, $\overrightarrow{OY}$. Let $\hat{i}$, $\hat{j}$ be unit vectors along $\overrightarrow{OX}$, $\overrightarrow{OY}$ respectively (Fig. 23), let θ be the angle between the two vectors $\vec{F}$ and $\hat{i}$.

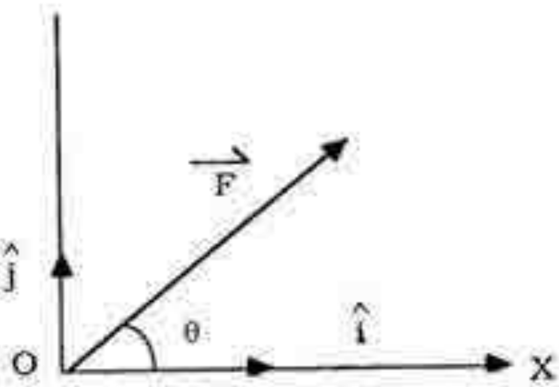


Fig. (23)

We know that the force $\vec{F}$ can be written in the form $\vec{F} = (F \cos \theta)\ \hat{i} + (F \sin \theta)\ \hat{j}$ where $\| \vec{F} \| = F$ but $\vec{F} \odot \hat{i} = F \cos \theta$

$\vec{F} \cdot \hat{j} = F \cos(90 - \theta) = F \sin\theta$ and thus we can write the vector $\vec{F}$ in the form $\vec{F} = (\vec{F} \odot \hat{i})\ \hat{i} + (\vec{F} \odot \hat{j})\ \hat{j}$.

This last relation shows that the component of the vector $\vec{F}$ in direction of the unit vector $\hat{i}$ is equal to $(\vec{F} \odot \hat{i})$. In general if we want to resolve the vector $\vec{F}$ in two perpendicular directions, one of which is the direction of the vector $\vec{a}$, then the component of the vector $\vec{F}$ in the direction of $\vec{a}$ is equal to the scalar product of $\vec{F}$ and a unit vector in the direction of $\vec{a}$.

But a unit vector in direction of $\vec{a}$ is $\dfrac{\vec{a}}{\|\vec{a}\|}$

The algebraic component of vector $\vec{F}$ in direction of vector $\vec{a}$ is equal to $\dfrac{\vec{F} \odot \vec{a}}{\|\vec{a}\|}$

Notice that the algebraic component of vector $\vec{F}$ in direction of vector $\vec{a}$ is equal to the algebraic projection of vector $\vec{F}$ in direction of vector $\vec{a}$.

Example (1) :

Determine the algebraic component of force $\vec{F} = 7\ \hat{i} + 24\ \hat{j}$ in direction of vector $\overrightarrow{AB}$, where $A = (1\ ,\ 2)$ and $B = (4\ ,\ 6)$.

Solution :

Referring to fig. 24 .

$\overrightarrow{OA} = \hat{i} + 2\hat{j}$, $\overrightarrow{OB} = 4\hat{i} + 6\hat{j}$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = 3\hat{i} + 4\hat{j}$

$\| \overrightarrow{AB} \| = \sqrt{9 + 16} = 5$

The algebraic component of $\vec{F}$ in direction of $\overrightarrow{AB}$ is equal to

$\vec{F} \odot \frac{\overrightarrow{AB}}{\|\overrightarrow{AB}\|}$

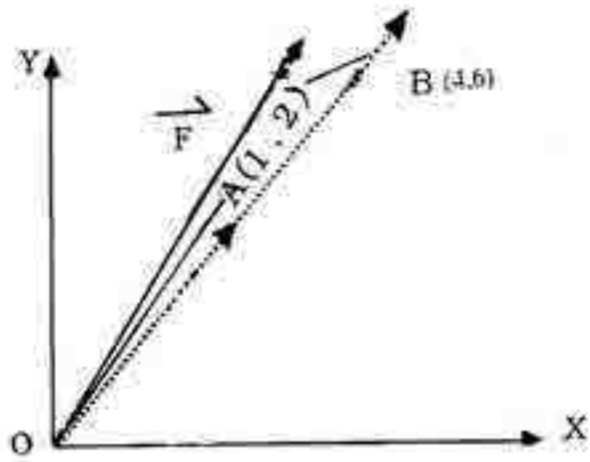


Fig. (24)

$$= (7\hat{i} + 24\hat{j}) \odot \frac{(3\hat{i} + 4\hat{j})}{5} = \frac{21 + 96}{5} = \frac{117}{5} = 23.4$$

II. Vector Product of Two Vectors :

Let $\vec{a}$ and $\vec{b}$ be two non-zero vectors, and $\| \vec{a} \| = a$, $\| \vec{b} \| = b$ and let θ be the angle between the two vectors when drawn outwards from the same point.

Definition :

The vector product of the two vectors $\vec{a}$ and $\vec{b}$, denoted by $\vec{a} \times \vec{b}$ is defined as $\boxed{\vec{a} \times \vec{b} = (ab \sin \theta)\, \hat{c}}$ where $\hat{c}$ is a unit vector perpendicular to the plane containing the two vectors $\vec{a}$, $\vec{b}$. The direction of the unit vector $\hat{c}$ is determined according to the right hand rule which states that "If the curved fingers of the right hand indicates the rotation of vector $\vec{a}$ towards $\vec{b}$ through the smaller angle, then the thumb will indicate the direction of $\hat{c}$ as in Figs. (25) and (26). If either $\vec{a}$ or $\vec{b}$ or both is a zero vector, then,

$\vec{a} \times \vec{o} = \vec{o} \times \vec{b} = \vec{o} \times \vec{o} = \vec{o}$. From the definition of vector product of two vectors we have the following two important properties.

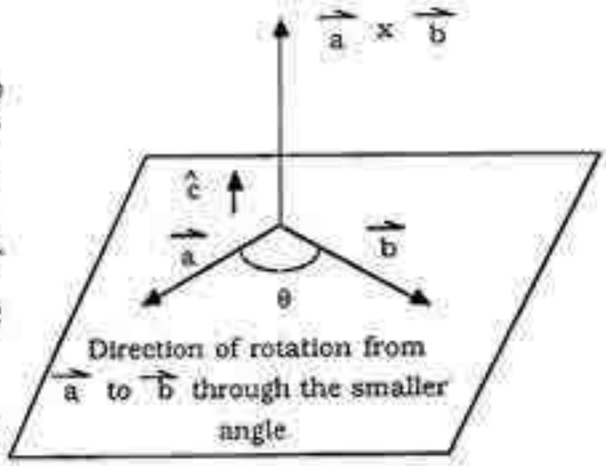


Fig. (25)

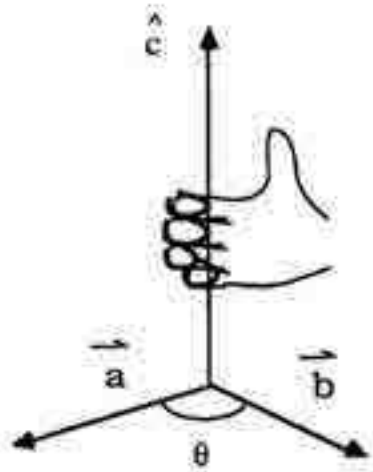


Fig. (26)

1. $\vec{b} \times \vec{a} = -(\vec{a} \times \vec{b})$

   This property can be proved easily if we notice that when we determine the direction of $\vec{b} \times \vec{a}$ the rotation will be from $\vec{b}$ to $\vec{a}$ through the smaller angle.

   Fig. (27).

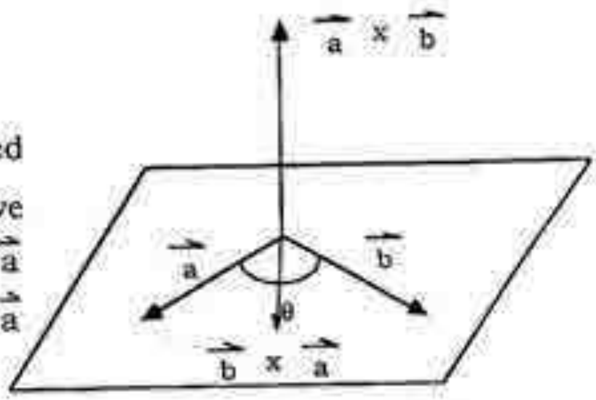


Fig. (27)

2. The vector product of two parallel vectos is the zero vector. If $\vec{a}$ and $\vec{b}$ are parallel vectors, then $\theta = 0$ Or 180°, and in both cases $\sin\theta = 0$. As a special case, the vector product of a vector multiplied by itself is a zero vector i.e. $\vec{a} \times \vec{a} = \vec{o}$.

There are also two properties to the vector product, of two vectors which we state in the following two theorems.

Theorem 1 :

For any two vectors $\vec{a}$ and $\vec{b}$

$(m\,\vec{a} \times \vec{b}) = (\vec{a} \times m\,\vec{b}) = m\,(\vec{a} \times \vec{b})$

where m is any scalar.

Proof :

If either $\vec{a}$ or $\vec{b}$ or both is a zero vector or if $m = 0$, the theorem follows directly from the definition of vector product of two vectors.

If neither $\vec{a}$ nor $\vec{b}$ is a zero vector and $m \neq 0$, there are two cases :

Case (1) : If $m > 0$

In this case the smaller angle between $m\,\vec{a}$ and $\vec{b}$ is the same as the smaller angle between $\vec{a}$ and $\vec{b}$ (Fig. 28).

(Refer to the corresponding theorem in the case of the scalar product of two vectors).

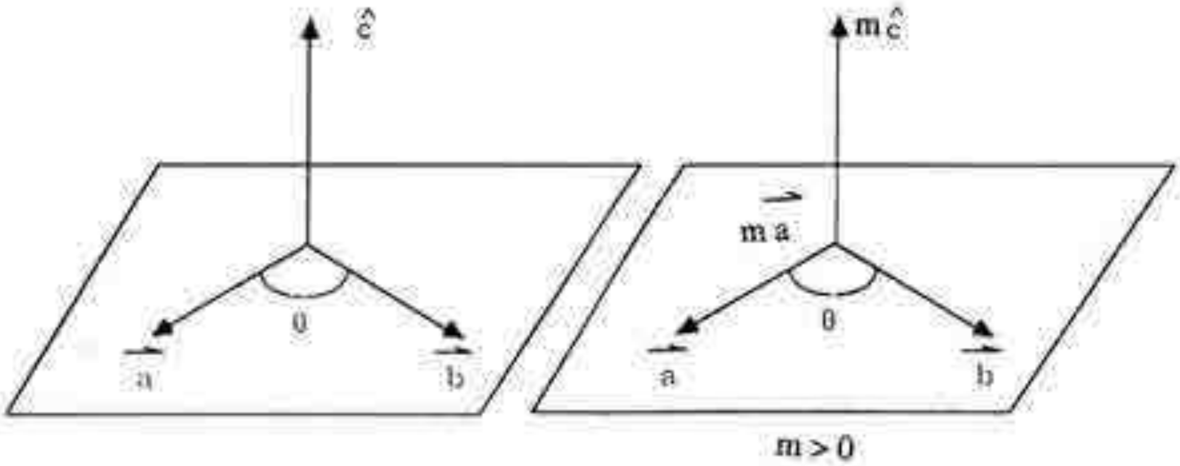


Fig. (28)

Therefore if $\vec{a} \times \vec{b} = (ab \sin \theta)\ \hat{c}$.

where θ is the angle between the vectors $\vec{a}$ and $\vec{b}$ and $\hat{c}$ is a unit vector perpendicular to the plane containing the two vectors $\vec{a}$ and $\vec{b}$ and whose direction is determined by the right hand rule.

$\therefore \quad (m\ \vec{a}) \times \vec{b} = (m\ a\ b \sin \theta)\ \hat{c} = m\ (\vec{a} \times \vec{b})$

To prove the second part of the theorem we have

$$\vec{a} \times (m\vec{b}) = -(m\ \vec{b} \times \vec{a}) = -m\,(\vec{b} \times \vec{a})$$

$$= m\,(\vec{a} \times \vec{b}).$$

Case (2) : If $m < 0$

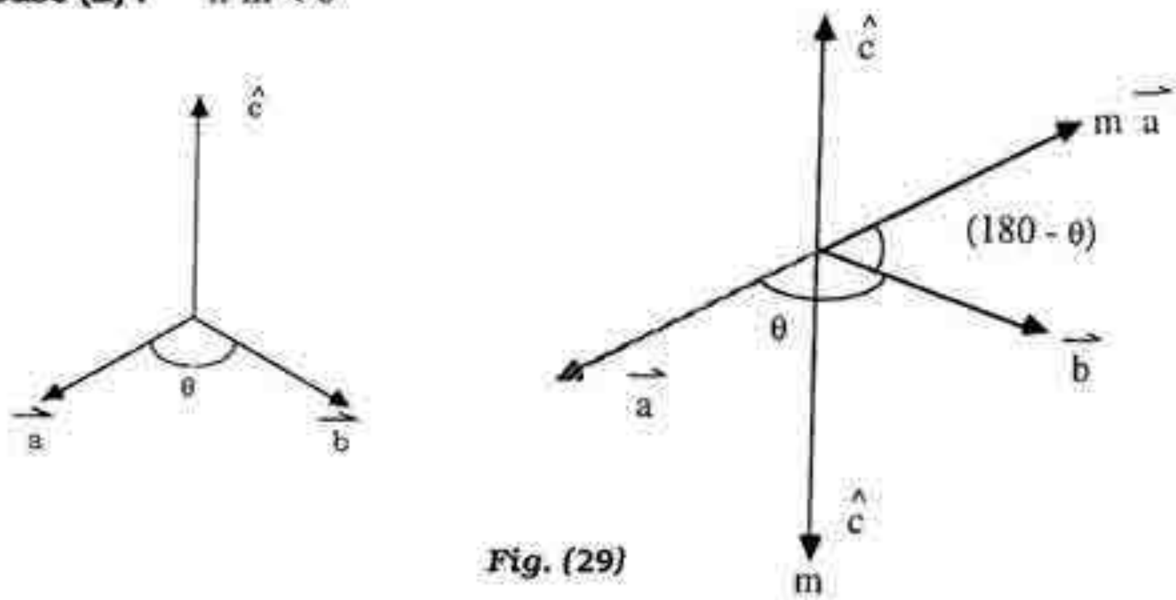


Fig. (29)

In this case the smaller angle between the vectors $m\,\vec{a}$ and $\vec{b}$ is $(180^\circ - \theta)$ and the direction of rotation from $m\,\vec{a}$ to $\vec{b}$ through the smaller angle between them is opposite to the direction of rotation from $\vec{a}$ to $\vec{b}$ through the smaller angle between them (Fig. 29).

$$\therefore \quad (m\,\vec{a}) \times \vec{b} = |\,m\,|\; a\,b \sin(180^\circ - \theta)\,(-\,\hat{c})$$
$$= -\,|\,m\,|\; a\,b \sin\theta\;\hat{c} = m\,(a\,b \sin\theta\,)\;\hat{c}$$
$$= m\,(\,\vec{a} \times \vec{b}\,)$$

Theorem 2 : without proof

For any three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ the distributive property is satisfied.

$$(\vec{a} + \vec{b}) \times \vec{c} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c}.$$

The Geometric meaning of the magnitude of the vector prouct of two vectors :

Consider two non-zero vectors $\vec{a}$ and $\vec{b}$ and let $a = \| \vec{a} \|$, $b = \| \vec{b} \|$, θ be the smaller angle between these two vectors when drawn from the same point.

From the definition $\| \vec{a} \times \vec{b} \| = a\, b \sin \theta$.

We notice that this quantity is equal to the area of the parallelogram drawn on the two straight portions representing the two vectors $\vec{a}$, $\vec{b}$ as two adjacent sides, or equal to twice the area of the triangle in which these two portions form two adjacent sides (Fig. 30).

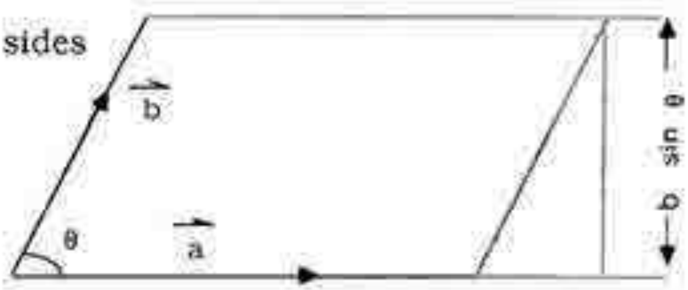


Fig. (30)

The right hand system of unit vectors :

Let $\overrightarrow{OX}$, $\overrightarrow{OY}$ be two perpen, dicular directions, and $\hat{i}$, $\hat{j}$ be unit vectors in these two directions respectively (Fig. 31). Draw $\overrightarrow{OZ}$ perpendicular to the plane containing $\overrightarrow{OX}$, $\overrightarrow{OY}$ let $\hat{k}$ be a unit vector in this direction. We choose the direction of $\overrightarrow{OZ}$ so that $\hat{k}$ satisfies the right hand rule when $\hat{i}$ is rotated towards $\hat{j}$ through the right angle between them at O (the smaller angle between $\hat{i}$ and $\hat{j}$)

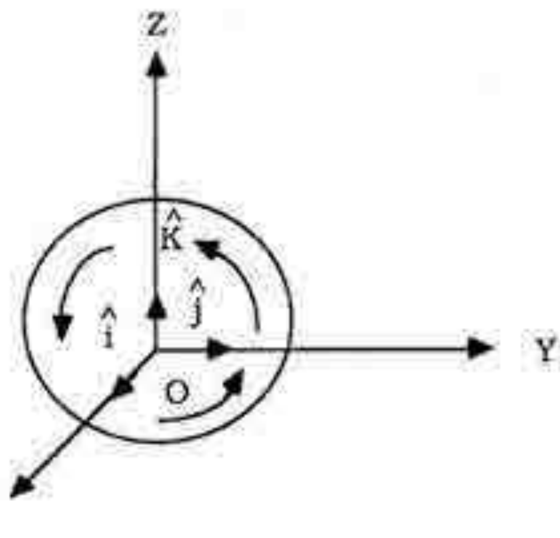


Fig. (31)

Referring to Fig. (31) it is clear that

$$\hat{i} \times \hat{j} = \hat{k}$$

also $\hat{j} \times \hat{k} = \hat{i}$

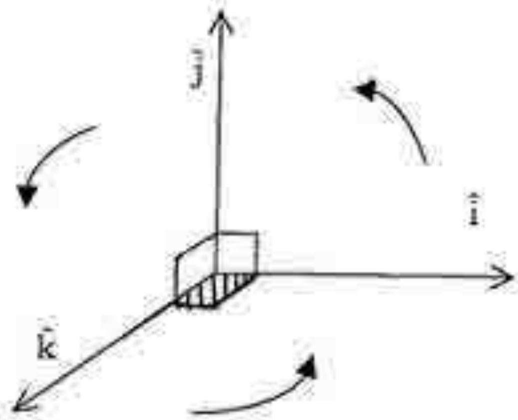
and $\hat{k} \times \hat{i} = \hat{j}$

(Notice the periodic order in the above three relations). Thus, we say that the system of unit vectors ($\hat{i}$, $\hat{j}$, $\hat{k}$) form a right hand system of unit vectors, notice the order of these vectors in the brackets and we say

The rotation from $\hat{i}$ to $\hat{j}$ through the smaller angle defines $\hat{k}$

The rotation from $\hat{j}$ to $\hat{k}$ through the smaller angle defines $\hat{i}$

The rotation from $\hat{k}$ to $\hat{i}$ through the smaller angle defines $\hat{j}$



This is denoted the periodic order.

The components of the vector product ($\vec{a} \times \vec{b}$) in rectangular resolution :

Let the two vectors $\vec{a}$, $\vec{b}$ be written in rectangular resolution as $\vec{a} = a_1 \hat{i} + a_2 \hat{j}$, $\vec{b} = b_1 \hat{i} + b_2 \hat{j}$

Using the above theorems and referring to Fig. (31)

$$\overrightarrow{a} \times \overrightarrow{b} = (a_1 \hat{i} + a_2 \hat{j}) \times b_1 \hat{i} + b_2 \hat{j})$$

$$= (a_1 b_1) \times (\hat{i} \times \hat{i}) + (a_1 b_2)(\hat{i} \times \hat{j}) + (a_2 b_1)(\hat{j} \times \hat{i})$$

$$+ (a_2 b_2)\ (\hat{j} \times \hat{j})$$

$$= (a_1 b_2)\ \hat{k} + (a_2 b_1)\ (-\hat{k})$$

$$= (a_1 b_2 - a_2 b_1)\ \hat{k}$$

Example (2) :

Assumming that $(\hat{i}, \hat{j}, \hat{k})$ form a right hand system, find the vector product of vector a into vector b where $\overrightarrow{a} = 3\hat{i} + 4\hat{j}$, $\overrightarrow{b} = 5\hat{i} + 12\hat{j}$ Find also the area of the triangle drawn on the two straight vector portions representing these two vectors as two adjacent sides.

Solution :

$$\overrightarrow{a} \times \overrightarrow{b} = (3 \times 12 - 4 \times 5)\ \hat{k}$$

$$= 16\ \hat{k}$$

$\therefore$ Area of triangle = $1/2 \times 16 = 8$ square units.

Exercises (2)

1. ABCD is a square, the length of whose side is 10 cm. Find the scalar product of the two vectors $\overrightarrow{BD}$, $\overrightarrow{BA}$. Calculate also the algebraic projection of vector $\overrightarrow{BD}$ in direction of vector $\overrightarrow{CB}$.

2. ABCD is a rectangle in which AB = 40cm, BC = 30cm. Find the algebraic projection of vector $\overrightarrow{BD}$ in the directions of the vectors $\overrightarrow{BC}$ and $\overrightarrow{DC}$.

3. It is required to resolve a force $\vec{F}$ into two components $\overrightarrow{F_1}$, $\overrightarrow{F_2}$. If $\overrightarrow{F_1}$ is parallel to a given vector $\vec{b}$, while $\overrightarrow{F_2}$ is perpendicular to $\vec{b}$, prove that :
$\overrightarrow{F_1} = \frac{(\vec{F} \odot \vec{b})}{b^2} \vec{b}$, hence find $\overrightarrow{F_2}$

4. Find a unit vector perpendicular to both the two vectors $\vec{a} = \hat{i} - \hat{j}$ and $\vec{b} = \hat{i} + \hat{j}$
(Hint : find the magnitude of the vector $\vec{a} \times \vec{b}$)

5. $\overrightarrow{F_1}$, $\overrightarrow{F_2}$, $\overrightarrow{F_3}$ are three force vectors which satisfy the relation $\overrightarrow{F_1} + \overrightarrow{F_2} + \overrightarrow{F_3} = \vec{0}$. Prove that $\overrightarrow{F_1} \times \overrightarrow{F_2} = \overrightarrow{F_2} \times \overrightarrow{F_3} = \overrightarrow{F_3} \times \overrightarrow{F_1}$.
Interpret this result geometrically.

6. If $\vec{a} = 15\hat{i} + 6\hat{j}$, $\vec{b} = 3\hat{i} + 5\hat{j}$, $\vec{c} = \hat{i} + \hat{j}$, find each of the vectors : $(\vec{a} + \vec{b}) \times \vec{c}$, $(\vec{a} \times \vec{b}) \odot \vec{c}$, $\vec{a} \odot (\vec{b} \times \vec{c})$, $(\vec{b} \times \vec{a}) \times \vec{c}$, $\vec{b} \times (\vec{a} \times \vec{c})$ and $\vec{a} \times (\vec{c} \times \vec{b})$.

7. Find the vector product $\vec{A} \times \vec{B}$ given that :
$\vec{A} = 5\hat{i} - 4\hat{j}$ and $\vec{B} = 3\hat{i} + 7\hat{j}$, then find the area of the triangle in which two adjacent sides are represented by the two directed line segments representing $\vec{A}$ and $\vec{B}$.

MOMENT OF A FORCE ABOUT A POINT :

Let A be any point on the line of action of the $\vec{F}$, $\vec{r}$ the position vector of A with respect to O. (Fig. 32).

Definition :

The moment of the force $\vec{F}$ with respect to or about O denoted by $\overrightarrow{M_o}$ is defined to be the vector quantity $\vec{r} \times \vec{F}$ i.e. $\overrightarrow{M_o} = \vec{r} \times \vec{F}$.

It is easy to notice that $\overrightarrow{M_o}$ is independent of the choice of point A on the line of action of the force $\vec{F}$.

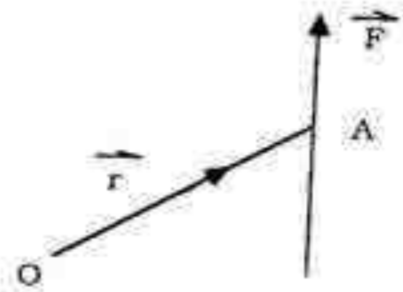


Fig. (32)

Proof: Let B be any other point on the line of action of the foce, $\overrightarrow{r_1}$ its position vector with respect to O (Fig. 33).

We have :

$$\overrightarrow{r_1} \times \vec{F} = (\vec{r} + \overrightarrow{AB}) \times \vec{F}$$
$$= \vec{r} \times \vec{F} + \overrightarrow{AB} \times \vec{F}$$

$\because$ $\overrightarrow{AB}$ is parallel to $\vec{F}$,

$\therefore$ $\overrightarrow{AB} \times \vec{F} = \vec{O}$

$\therefore$ $\overrightarrow{r_1} \times \vec{F} = \vec{r} \times \vec{F}$

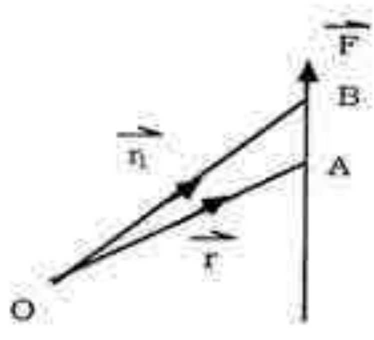


Fig. (33)

Therefore the moment of the force $\vec{F}$ about O is independent of the position of the point chosen on the line of action of the force when calculating the moment. From the definition it is clear that the moment of the (non-zero) force $\vec{F}$ vanishes when $\vec{r} = \vec{o}$, or if $\vec{r}$ is parallel to $\vec{F}$, i.e. if the line of action of the force passes through the point about which we take the moment.

The angle between $\vec{r}$, $\vec{F}$:

When calculating the moment of the force $\vec{F}$ about a point, it is very important that the student can determine the angle θ between the two vectors $\vec{r}$ and $\vec{F}$ in a proper way.

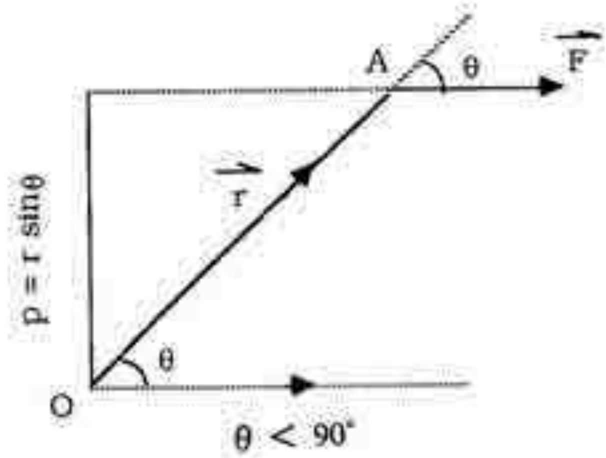


Fig. (34 - b)

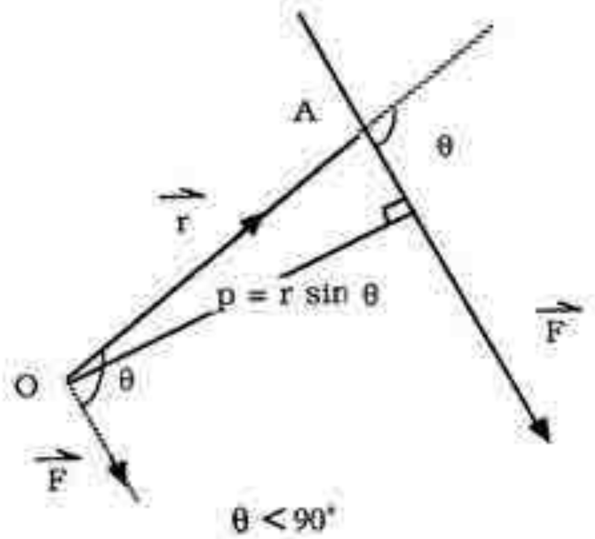


Fig. (34 - a)

The student has to remember that to determine θ, the two vectors $\vec{r}$ and $\vec{F}$ must be drawn outwards from the same point. Therefore we have to imagine that the force vector $\vec{F}$ is transferred parallel to itself to pass through O.

Figures (34) show two different cases to determine the angle between $\vec{r}$ and $\vec{F}$.

It is easy to notice that in all cases the length of the normal from O to the line of action of the force $\vec{F}$ is given by the relation.

$p = r \sin \theta$ where $r = \| \vec{r} \|$

Magnitude and direction of the moment vector of a force about a point :

Let $F = \| \vec{F} \|$, p the length of the perpendicular from the point O about which we calculate the moment, to the line of action of the force $\vec{F}$.

The magnitude of the vector moment of the force $\vec{F}$ about O is given by the relation $\| \overrightarrow{M_o} \| = \| \vec{r} \times \vec{F} \| = r F \sin \theta$

$$= F (r \sin \theta) = F p$$

$$\boxed{\therefore \| \overrightarrow{M_o} \| = F p}$$

The unit of the magnitude of moment is equal to the product of a unit of length times the unit of magnitude of a force. The direction of the vector moment $\overrightarrow{M_o}$ is that of a unit vector $\hat{c}$ which is perpendicular to the plane containing both the vectors $\vec{r}$ and $\vec{F}$, its direction is determined by the right hand rule, previously explained when the vector $\vec{r}$ is rotated towards the vector $\vec{F}$ (after transferring the starting point of $\vec{F}$ to O) through the smaller angle between them. (Fig. 35).

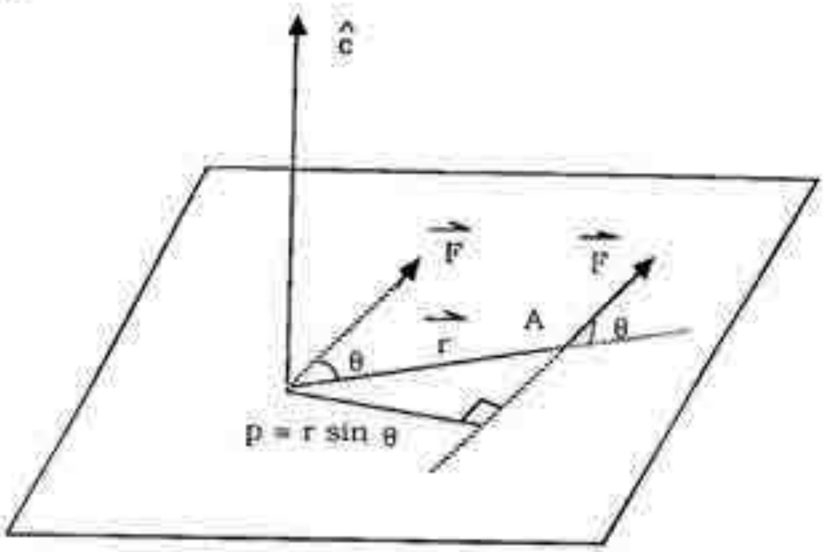


Fig. (35)

Plane containing both $\vec{r}$ and $\vec{F}$

$$\boxed{\vec{M}_o = (FP)\ \hat{c}}$$

Example (1) :

$\hat{i}$ and $\hat{j}$ are two unit vectors in the two perpendicular directions $\overrightarrow{OX}$, $\overrightarrow{OY}$ respectively. A force $\vec{F} = 2\,\hat{i} - 3\,\hat{j}$ acts at the point $A = (1\ ,\ 2)$.

Calculate the moment of this force about O, and find the length of the perpendicular from O on the line of action of the force.

Solution :

Let $\hat{k}$ be a unit vector perpendicular to both $\hat{i}$ and $\hat{j}$ such that [$\hat{i}$, $\hat{j}$, $\hat{k}$] form a right hand system of unit vectors (Fig. 36).

We have $\vec{r} = \overrightarrow{OA} = \hat{i} + 2\hat{j}$

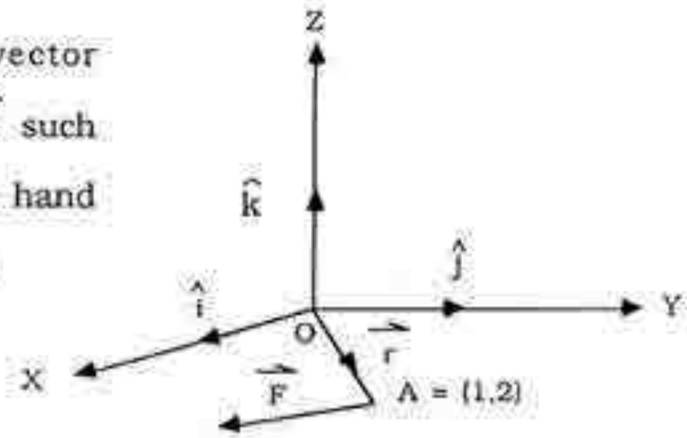


Fig. (36)

$\therefore \quad \overrightarrow{M_o} = \vec{r} \times \vec{F}$

$= \quad (\hat{i} + 2\hat{j}) \times (2\hat{i} - 3\hat{j})$

$= \quad [1 \times (-3) - 2 \times 2] \overrightarrow{M_0} = -7\hat{k}$

We can calculate the length of the perpendicular from the origin O on the line of action of the force $\vec{F}$

$\therefore \quad \| \overrightarrow{M_o} \| = F\,p$

$$\therefore \quad P = \frac{\|\overrightarrow{M_o}\|}{F} = \frac{7}{\sqrt{(2)^2 + (-3)^2}}$$

$$= \frac{7}{\sqrt{13}} \text{ units of length}$$

MOMENT OF COPLANAR FORCES

In what follows we shall be concerned only with a system of coplanar forces, i.e.a system of forces whose lines of action lie in the same plane.

In this case the calculation of the moments of these forces about a point in its plane will be much easier than before. Let $\vec{F}$ be any force of the system. $\vec{r}$ the position vector of an arbitrary point A on its line of action with respect to the point O lying in the place of the forces and about which we are taking moments, $F = \|\vec{F}\|$. P the length of the perpendicular from O on the line of action of the force.

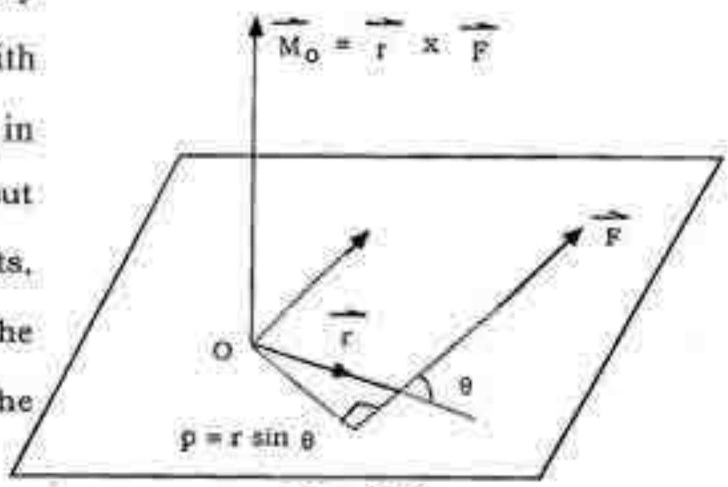


Fig. (37)

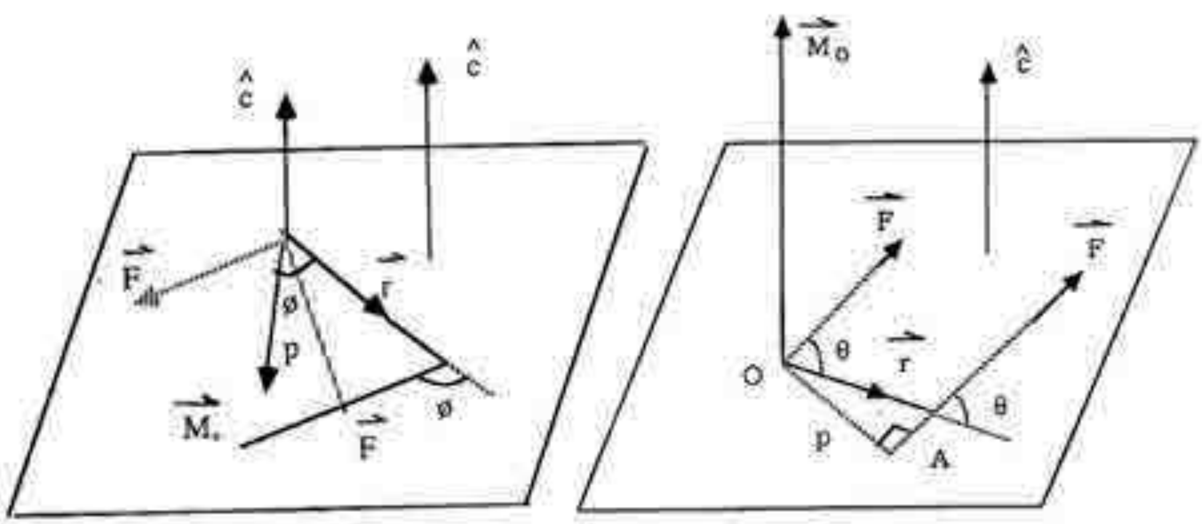
Since the two vectors $\vec{F}$ and $\vec{r}$ lie in the plane of the forces. The moment vector $\vec{M_o} = \vec{r} \times \vec{F}$ will be perpendicular to this plane as in Fig. (37), its magnitude is equal to the product of the magnitude of the force times the length of the perpendicular from O on its line of action. Thus in the case of a system of coplanar forces, all moment vectors will be parallel and perpendicular to the plane of the forces.

This enables us to use the algebraic measures of the moment vectors instead of the moment unit vectors themselves.

$$\therefore \overrightarrow{M_o} = M_o \hat{c}$$

where M_o is the algebraic measure of the moment vector $\overrightarrow{M_o}$ with respect to the unit vector $\hat{c}$ which is perpendicular to the plane of the forces.

Figures (38) shows the different possible cases.



$\overrightarrow{M_o}$ in a direction opposite to $M_o \hat{c} = (-Fp) \hat{c}$

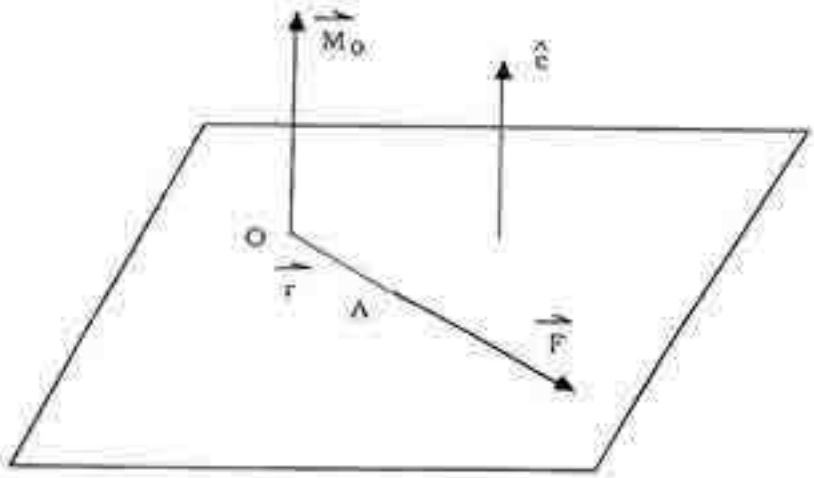
$M_o = -Fp$

Fig. (38 b)

$\overrightarrow{M_o}$ in the same direction as $M_o \hat{c} = (Fp) \hat{c}$

$M_o = Fp$

Fig. (38 a)



$\vec{M}_O = \vec{O}$, $M_O = O$ (the line of action of the force passes throught O)

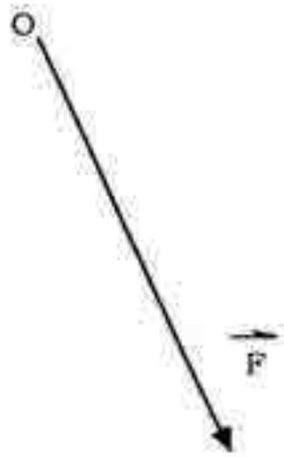
Fig. (38 - c)

In what follows we are not going to mention the unit vector $\hat{c}$ since we are going to deal only with the algebraic measures of the moment vectors, the sign of the algebraic measure M_o will be defined according to the following rule.

Figs (39).

Rule :

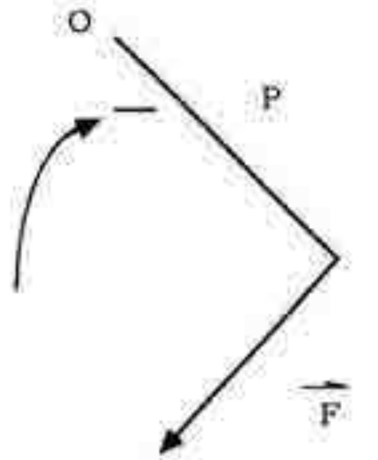
If when looking at the plane of the forces, we find that the force is tending to rotate the body about O in an anti-clockwise direction, the algebraic measure of the moment vector is considered positive as in fig. (39 a), but if the force is tending to rotate the body about O in a clockwise direction, the algebraic measure of the moment vector is considered negative as in fig. (39 b), if the line of action of the force passes through O, the algebraic measure of the moment vector is zero as in fig. (39 c).



line of action of the force passes through O

$M_O = O$

Fig. (39 -c)



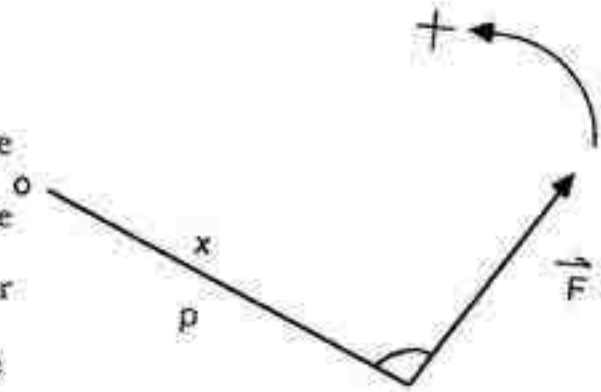
rotation in a clockwise direction $M_o = -Fp$

Fig. (39 - b)

The length of the perpendicular from O to the line of acton of the force is called the "arm of the force"

N.B.

In what follows when we mention the algebraic sum of the moments of a system of coplanar forces about a point we mean the algebraic sum of the algebraic measures of the moment vectors of these forces about this point taking into consideration the above rule.



rotation in an anti-clockwise direction

$M_o = Fp$

Fig. (39 a)

Example (1) :

ABCD is a square, the length of whose side is L. Forces of magnitudes F, 2 F, 3 F, 4 F, $8\sqrt{2}$ F, units of force act along $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{CD}$, $\overrightarrow{DA}$, $\overrightarrow{CA}$ respectively. Calculate the algebraic sum of the moments of these forces about the vertex B.

Solution :

Referring to fig. (40), we notice that the two forces F and 2F pass through B, and thus their moment about B vanish. The arm of the force 3 F with respect to B is $\overline{CB}$, this force is tending to rotate about B in a clockwise direction, i.e. the magnitude of its moment about B is negative and equal to $-3\text{ F} \times BC = -3\text{ F L}$.

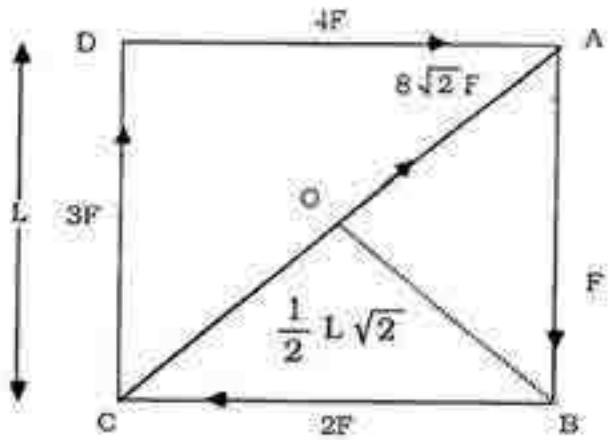


Fig. (40)

The arm of the force 4 F with respect to B is $\overline{AB}$, this force tends to rotate about B in a clockwise direction, and thus the magnitude of its moment is equal to $-4\text{ F} \times AB = -4\text{ F L}$.

Finally we notice that the arm of the force $8\sqrt{2}$ F is $\overline{OB}$ whose length is $1/2\text{ L}\sqrt{2}$ where O is the centre of the square. This force tends to rotate about B in a clockwise direction, i.e. its moment is negative and equal to.

$$-(8\sqrt{2}\,F) \times (1/2\,L\sqrt{2}) = -8\,F\,L$$

$\therefore$ The algebraic sum of the moments of the forces about B

$$= -3\,F\,L - 4\,F\,L - 8\,F\,L = -15\,F\,L$$

Example (2) :

A B C D is a rectnagle in which AB = 6cm, BC = 8cm, $E \in \overline{BC}$ where BE = 3cm. Forces of magnitudes 9, 12, 15, 17, 10, $6\sqrt{5}$ newtons acted along $\overrightarrow{AB}$, $\overrightarrow{CB}$, $\overrightarrow{CD}$, $\overrightarrow{AD}$, $\overrightarrow{AC}$ and $\overrightarrow{AE}$ respectively. Find the algebraic sum of the moments of these forces about each of the points A, B, C, E, and M where M is the intersection of the diagonals.

Solution :

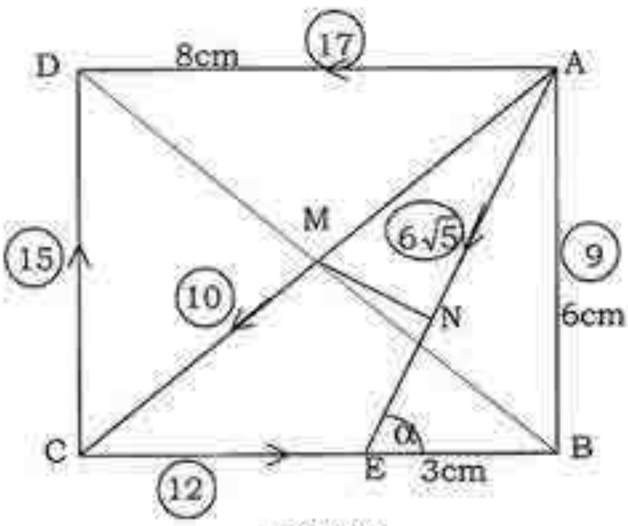


Fig. (41)

In the triangle ABC :

$$\frac{AB}{AC} = \frac{6}{10} = \frac{3}{5} \text{ and } \frac{3}{5} = \frac{BE}{EC}$$

$\therefore \overrightarrow{AE}$ bisects $B\hat{A}C$.

In the triangle AMN, $m(\hat{N}) = 90°$

$$\begin{aligned} \therefore \quad MN &= AM \sin M\hat{A}N \\ &= 5 \sin M\hat{A}N \\ &= 5 \sin E\hat{A}B \\ &= 5 \times \frac{3}{3\sqrt{5}} = \sqrt{5} \end{aligned}$$

$$AE = \sqrt{9+36} = 3\sqrt{5} \text{ cm}$$

$$AC = BD = \sqrt{36+64} = 10\text{cm}$$

$$M_A = 12 \times 6 - 15 \times 8 = 72 - 120 = -48 \text{ N.cm}$$

$$\begin{aligned} M_B &= -15 \times 8 + 17 \times 6 + 6\sqrt{5} \times 3 \sin\theta + 10 \sin\alpha \\ &= -120 + 102 + 6\sqrt{5} \times \frac{6}{3\sqrt{5}} + 10 \times 8 \times \frac{6}{10} \\ &= -120 + 102 + 12 + 42 = 66 \text{ N.cm} \end{aligned}$$

$$M_C = 17 \times 6 - 9 \times 8 - 6\sqrt{5} \times \frac{6}{3\sqrt{5}} \times 5 = 102 - 72 - 60 = -30 \text{ N.cm}$$

$$M_E = 17 \times 6 - 15 \times 5 - 9 \times 3 + 10 \times 5 \times \frac{6}{10}$$
$$= 102 - 102 + 30 = 30 \text{ N.cm}$$
$$M_M = 17 \times 3 + 12 \times 3 - 15 \times 4 - 9 \times 4 - 6\sqrt{5} \times \frac{1}{2} \times 5 \times \frac{6}{3\sqrt{5}}$$
$$= 51 + 36 - 60 - 36 - 30 = -39 \text{ N.cm.}$$

Example (3) :

A rod AB of length 100 cm is bent at its midpoint O so that $\overline{AO}$ is perpendicular to $\overline{OB}$. Forces of magnitudes 10, 20, $30\sqrt{2}$ kg.wt act A and B as shown in fig. 42 what is the magnitude of the force $\vec{F}$ which should act at the midpoint of $\overline{OB}$ in the shown direction so that the algebraic sum of the moments about O may vanish ?

Solution :

The arm of the force $30\sqrt{2}$ kg. wt. is $\overline{OC}$ which is the perpendicular from O to the line of action of the force.

Fig. (42) Fro the figure.

$OC = 50 \times \frac{1}{\sqrt{2}}$

$= 25\sqrt{2}$ cm.

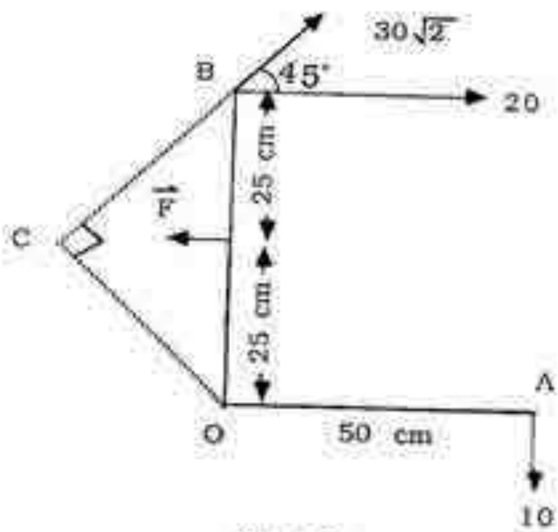


Fig. (42)

The algebraic sum of this moments about O

$= -10 \times 50 + 25\,F - 20 \times 50 -$

$(30\sqrt{2})\,(25\sqrt{2}) = 0$

$\therefore\ M_0 = O \qquad \therefore\ 25F = 3000$

i.e. $F = \dfrac{3000}{25} = 120$ kg. wt.

Example (4) :

Three forces $\overrightarrow{F_1} = \hat{i} + \hat{j}$, $\overrightarrow{F_2} = \hat{i} - \hat{j}$, $\overrightarrow{F_3} = 2\,\hat{i} - 3\,\hat{j}$ act at the point A =(1 , 1). Find the moment of each force about the point B =(3 , - 1), then calculate the sum of these moments. Find also the resultant of the three forces, then find its moment about B. What do you deduce when comparing the results ?

Solution :

Let $\overrightarrow{M_1}$, $\overrightarrow{M_2}$ and $\overrightarrow{M_3}$ be the moments of the three forces about point B and let $\vec{r} = \overrightarrow{BA}$.

$\overrightarrow{M_1} = \vec{r} \times \overrightarrow{F_1} = (-2\,\hat{i} + 2\,\hat{j}) \times (\hat{i} + \hat{j})$

$= (-2 \times 1 - 1 \times 2)\ \hat{k} = -4\ \hat{k}$

$\overrightarrow{M_2} = \vec{r} \times \overrightarrow{F_2} = (-2\,\hat{i} + 2\,\hat{j}) \times (\hat{i} - \hat{j})$

$= -2\,(\hat{i} - \hat{j}) \times (\hat{i} - \hat{j})$

$= -2 \times \vec{0} = \vec{0}$

$\overrightarrow{M_3} = \vec{r} \times \overrightarrow{F_3} = (-2\,\hat{i} + 2\,\hat{j}) \times (2\,\hat{i} - 3\,\hat{j})$

$= (-2 \times -3 - 2 \times 2) = 2\ \hat{k}$

$$\overrightarrow{M_1} + \overrightarrow{M_2} + \overrightarrow{M_3} = -4\,\hat{k} + \overline{o} + 2\,\hat{k} = -2\,\hat{k}$$

$$\vec{R} = \overrightarrow{F_1} + \overrightarrow{F_2} + \overrightarrow{F_3}$$

$$\vec{R} = (\hat{i} + \hat{j}) + (\hat{i} - \hat{j}) + 2\,\hat{i} - 3\,\hat{j}$$

$$\vec{R} = 4\,\hat{i} - 3\,\hat{j}$$

Moment of the resultant $= \vec{r} \times \vec{R}$

$$= (-2\,\hat{i} + 2\,\hat{j}) \times (4\,\hat{i} - 3\,\hat{j})$$

$$= (-2 \times -3 - 4 \times 2)\,\hat{k}$$

$$= -2\,\hat{k}$$

Noticing that the algebraic sum of the moments of the three forces about the point B is equal to the moment of the resultant of these forces about the same point.

Theorem :

The algebraic sum of the moments of a system of forces acting at a point about any point in space is equal to the moment of the resultant of these forces about the same point.

Proof :

Let $\overrightarrow{F_1}$, $\overrightarrow{F_2}$, $\overrightarrow{F_3}$, ... $\overrightarrow{F_n}$ be a finite number of forces acting at A and let O be any point in space Fig. (43).

We know that the resultant $\vec{R}$ of this system of forces passes also through A. To calculate the sum of the moments of the system about O. we have :

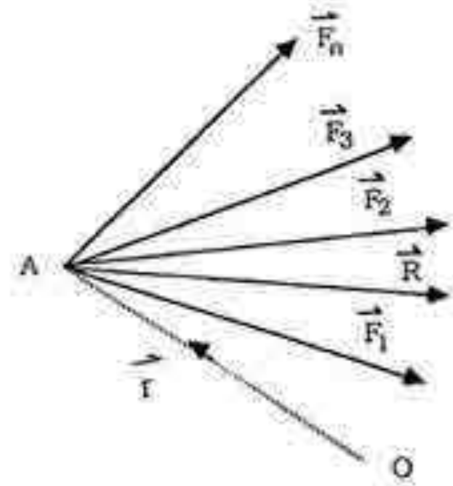


Fig. (43)

$$\overrightarrow{M_o} = \vec{r} \times \overrightarrow{F_1} + \vec{r} \times \overrightarrow{F_2} + \vec{r} \times \overrightarrow{F_3} + \dots\dots + \vec{r} \times \overrightarrow{F_n}$$

$$= \vec{r} \times (\overrightarrow{F_1} + \overrightarrow{F_2} + \overrightarrow{F_3} + \dots + \overrightarrow{F_n})$$

$$= \vec{r} \times \vec{R}$$

But this is the moment of the resultant $\vec{R}$ about O. since the resultant passes through A. and this proves the theorem.

Example (5) :

The forces $\overrightarrow{F_1} = \hat{i} + \hat{j}$, $\overrightarrow{F_2} = -\hat{i} + 2\hat{j}$, $\overrightarrow{F_3} = 3\hat{i} - \hat{j}$ act at the point A = (1 , 1). Find the moment of each of these forces about the origin, and hence find the length of the perpendicular from the origin to the line of action of the resultant.

Solution :

Let $\overrightarrow{M_1}$, $\overrightarrow{M_2}$, $\overrightarrow{M_3}$ be the moments of the three forces about the origin and let $\vec{r} = \overrightarrow{OA}$

$\overrightarrow{M_1} = \vec{r} \times \overrightarrow{F_1} = (\hat{i} + \hat{j}) \times (\hat{i} + \hat{j}) = \vec{O}$

$\overrightarrow{M_2} = \vec{r} \times \overrightarrow{F_2} = (\hat{i} + \hat{j}) \times (-\hat{i} + 2\hat{j})$

$= (1 \times 2 - 1 \times (-1)) \hat{k} = 3\hat{k}$

$\overrightarrow{M_3} = \vec{r} \times \overrightarrow{F_3} = (\hat{i} + \hat{j}) \times (3\hat{i} - \hat{j})$

$= (1 \times (-1) - 1 \times 3) \hat{k} = -4\hat{k}$

Moment of the resultant $= \vec{r} \times \vec{R}$

$= \overrightarrow{M_1} + \overrightarrow{M_2} + \overrightarrow{M_3}$

$= \vec{O} + 3\hat{k} - 4\hat{k} = -\hat{k}$

$\therefore \| \vec{r} \times \vec{R} \| = 1$

Also $\vec{R} = \overrightarrow{F_1} + \overrightarrow{F_2} + \overrightarrow{F_3} = 3\hat{i} + 2\hat{j}$

$\| \vec{R} \| = \sqrt{(3)^2 + (2)^2} = \sqrt{13}$

Therefore the length of the perpendicular

$= \dfrac{\| \vec{r} \times \vec{R} \|}{\| \vec{R} \|} = \dfrac{1}{\sqrt{13}}$ unit of length

Remarks :

1. If the sum of the moments of a system of forces about the point A = the sum of the moments of these forces about the point B, then the line of action of the resultant is parallel to $\overleftrightarrow{AB}$.
2. If the sum of the moments of a system of coplanar forces about the point A = - the sum of the moments of these forces about point B then the line of action of the resultant passes through the midpoint of $\overline{AB}$.

The General Theorem of Moments

"If the set of forces acting on a rigid body has a resultant, then the algebraic sum of the moments of these forces about a certain point is equal to the moment of the resultant about this point."

Results :

1. The algebraic sum of the moments of a set of forces about a point on the line of action of the resultant is equal to zero.
2. If the algebraic sum of a set of forces about a point vanishes, then either the resultant equals zero or its line of action passes through this point.

These two results have great importance in determining the line of action of the resultant of a set of coplanar forces.

Example (6) :

ABCD is a square of side length 6cm. Forces of magnitudes 1, 2, 3, 4 and F newtons act in $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{CD}$, $\overrightarrow{DA}$ and $\overrightarrow{AC}$ respectively.

If the line of action of the resultant passes through point $E \in \overline{BC}$ where BE = 1cm, find F.

Solution :

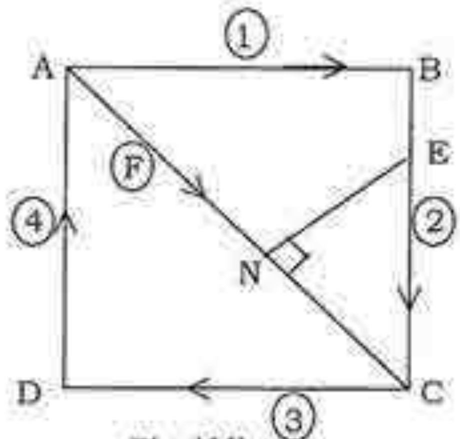


Fig. (44)

$EN = 5 \sin 45° = \frac{5\sqrt{2}}{2}$ cm.

$\because$ The line of action of the resultant passes through E $\therefore$ M = 0

$\therefore 1 \times 1 - 3 \times 5 - 4 \times 6 + F \times \frac{5\sqrt{2}}{2} = 0$

$\therefore \frac{5\sqrt{2}}{2} . F = 40$ $\qquad \therefore F = \frac{80}{5\sqrt{2}} = \frac{16}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$

$\therefore F = 8\sqrt{2}$ newtons.

Exercises (3)

In the following problems $\hat{i}$ and $\hat{j}$ are two unit vectors at right angles to each other, in the two directions $\overrightarrow{OX}$, $\overrightarrow{OY}$ respectively, while $\hat{k}$ is a unit vector perpendicular to both $\overrightarrow{OX}$, $\overrightarrow{OY}$ and such that $(\hat{i}, \hat{j}, \hat{k})$ form a right hand side system of unit vectors.

1. Find the moment vector of the force $\vec{F} = -\hat{i} + 3\hat{j}$ which acts through the point $A = (1, 2)$ about the origin $O = (0 , 0)$. Calculate the length of the perpendicular from O to the line of action of the force.

2. Find the moment of the force $\vec{F} = 3\hat{i} + 2\hat{j}$ which acts through the point $A (2, -1)$ about the point $B = (1 , 2)$, hence calculate the length of the perpendicular from B to the line of action of the force.

3. Find the moment of the force $\vec{F} = \hat{i} + \hat{j}$ which passes through the point $A = (-3 , -3)$ about the origin. Explain the result you obtain.

4. The two forces $\overrightarrow{F_1} = \hat{i} + \hat{j}$ $\overrightarrow{F_2} = m\hat{i} - 2\hat{j}$ act at the points $A_1 = (2,0)$, $A_2 = (0,2)$ respectively. Find the value of m if the sum of the moments about the origin vanishes.

5. The two forces $\overrightarrow{F_1} = m\hat{i} + 2\hat{j}$, $\overrightarrow{F_2} = n\hat{i} - \hat{j}$ act at the points $A_1 = (1,1)$, $A_2 = (-1, -2)$ respectively. Find the values of the constants m, n, if the sum of their moments about each of the origin and about the point $B = (2,3)$ vanishes.

6. The forces $\overrightarrow{F_1} = 2\hat{i} - \hat{j}$, $\overrightarrow{F_2} = 5\hat{i} + 2\hat{j}$, $\overrightarrow{F_3} = -3\hat{i} + 2\hat{j}$, act at the points A = (1,1). Prove using the moments that the line of action of the resultant is parallel to the straight line passing through the two points B = (2,1) and C = (6,4).

7. ABC is an equilateral triangle, the length whose side is 2 cm. Forces of magnitudes 100, 200, 300 newtons act along $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{AC}$ respectively. Find the algebraic sum of the moments of these forces about : 1st the point of intersection of the normals to the triangle .

2nd the midpoint of $\overrightarrow{BC}$.

8. Three forces, $\overrightarrow{F_1} = 3\hat{i} + 12\hat{j}$, $\overrightarrow{F_2} = 9\hat{i} + 4\hat{j}$, $\overrightarrow{F_3} = 8\hat{i} + 14\hat{j}$, act at the origin O = (0.0). Find the moment of each force about the point B = (1,2), then calculate the sum of these moments. Find also the resultant of th three forces, then find its moment about B. What do you deduce when comparing the results ?

9. ABC is a triangle in which AB = 3cm, BC = 4cm, CA = 5cm. Forces of magnitudes 5, 10, 15 newtons act along $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{CA}$ respectively. Find the algebraic sum of the moments of these forces about A , B , C.

10. ABCD is a rhombus whose side is 12cm, m ($\angle$ A) = 60° Forces of magnitudes 11 , 6 , 5 , 7 newtons act along $\overrightarrow{BA}$, $\overrightarrow{BC}$, $\overrightarrow{DC}$, $\overrightarrow{DB}$ respectively. Find the algebraic sum of th moments of these forces :

1st about A.

2nd about the point of intersection of the diagonals.

11. ABCDEH is a regular hexagon whose side is 10cm long. Forces 3, 4 , 5 , 6 , 7 , 8 newtons act along $\overrightarrow{AB}$, $\overrightarrow{CB}$, $\overrightarrow{DC}$, $\overrightarrow{DE}$, $\overrightarrow{EH}$, $\overrightarrow{HA}$ respectively. Find the algebraic sum of the moments of these forces :

1st about A.

2nd about the centre of the hexagon.

12. ABC is a right-angled triangle at B, where BC = 6cm. m ($\angle$ A) = 60° Two forces 6 , 4 newtons act along $\overrightarrow{BA}$, $\overrightarrow{CA}$ respectively. Find point D $\in$ $\overline{BC}$ so that the algebraic sum of the moments about D = 0.

13. ABCD is a rectangle in which AB = 8cm, BC = 12cm. Forces of magnitudes 16 , 14, L, N gm.wt act along $\overrightarrow{AB}$, $\overrightarrow{CB}$, $\overrightarrow{CD}$, $\overrightarrow{AD}$ respectively. If the algebraic sum of the moments of these forces about C and also about the centre of the rectangle equal zero, Find L and N.

14. AB is a rod of length 60cm Forces of magnitudes 6, 8, $2\sqrt{2}$ 12 kg.wt act on the rod as shown in fig. (45). O is the midpoint of the rod. C is a point on the rod 15cm distant from the end B. Find the algebraic sum of the moments of this system of forces about C.

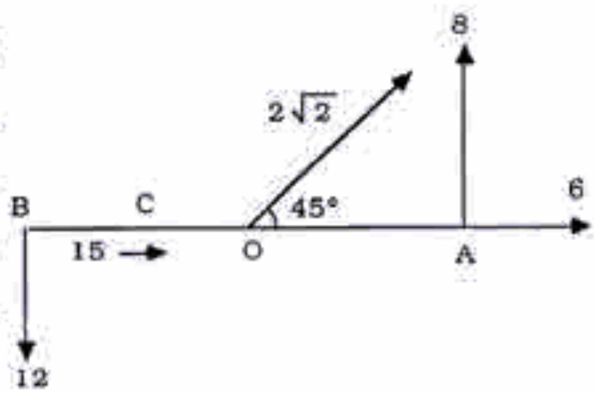


Fig. (45)

15. AB is a rod of length 120cm and neglected weight. Four forces of magnitudes 200, 200, 300, 100 newtons act on the rod at A, C, D, B, respectively, where C and D are the points of trisection of AB.

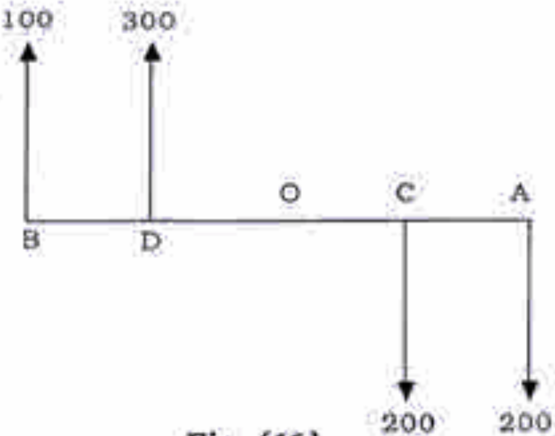


Fig. (46)

If all the forces are normal to the rod, and in the shown directions in fig. (46) find the algebraic sum of the moments of these forces about A and B, and O the midpoint of the rod. Compare the results you obtain.

16. ABC is a right-angled triangle at B, AB = 6cm, BC = 8cm. A force $\vec{F}$ acts in the plane of the triangle such that $M_A = M_B = 60$ newton cm, $M_C = -60$ newton cm. Find $\vec{F}$ and determine its action straight line.

17. Five forces of magnitudes 100, 200, 300, $\frac{200}{\sqrt{3}}$, $400\sqrt{2}$ newtons act in the square ABCD, the length of whose side is 1m as shown on fig. (47). Find the algebraic sum of the moments of these forces about the vertex C. What is the force which should act at the midpoint of $\overline{CD}$ in a direction perpendicular to this side so that the algebraic sum of the moments about C vanishes?

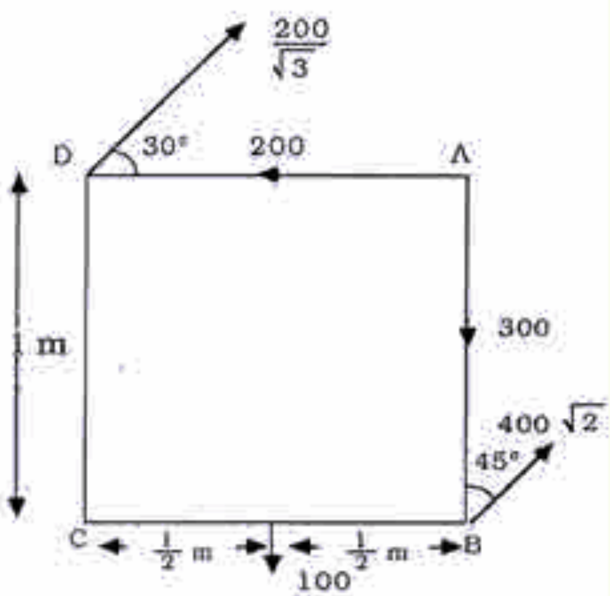


Fig. (47)

Chapter three

Parallel coplanar forces

Preface :

In this chapter we will deal with parallel coplanar forces.

Objectives:

By the end of teaching this chapter, the student should be able to:

(1) Recognize the resultant of two parallel forces.

(2) Determine the resultant of two parallel forces, its line of action and its direction.

(3) Find the resultant of a set of parallel forces, its point of action and its direction.

(4) Recognize the condition of equilibrium of a set of parallel coplanar forces.

Topics :

(1) The resultant of two parallel forces having the same direction.

(2) The resultant of two parallel forces having opposite direction.

(3) Moments of parallel coplanar forces.

(4) Equilibrium of parallel coplanar forces.

Parallel Coplanar Forces

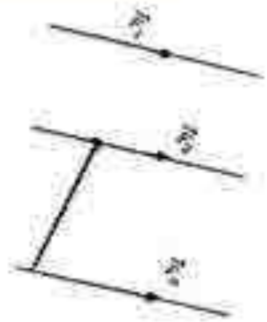
We have previously studied systems of forces acting on a particle. The lines of action of these forces meet, of course at this particle. We have also shown that the line of action of the resultant of such system passes through the common point of intersection of the lines of action of the forces, i.e. it passes through the particle.

We are going now to study a system of forces acting on a rigid body, the lines of action of which may not necessarily meet in a point.

In this chapter we are going to deal only with a system of forces all of whose lines of action are parallel and lie in the same plane, which we call. "parallel coplanar forces".

Let $\overrightarrow{F_1}$, $\overrightarrow{F_2}$, ... , $\overrightarrow{F_n}$ be a finite system of coplanar parallel forces as in fig. (48), and let $\overrightarrow{R}$ be its resultant.

In what follows we are going to assume that this resultant does not vanish, and we shall discuss how to find its magnitude, direction and line of action.



We have

$$\overrightarrow{R} = \overrightarrow{F_1} + \overrightarrow{F_2} + \dots\dots\dots + \overrightarrow{F_n} \qquad (1)$$

Fig. (48)

This relation determines the magnitude and direction of the resultant. We notice that it is parallel to the system of forces.

Therefore, it remains how to determine the line of action of the resultant, which we will explain later using the following theorem.

Theorem :

If ABC is a given triangle. m & n are two real numbers, then :

a) $m\ \overrightarrow{AB} + n\ \overrightarrow{AC} = (m + n)\ \overrightarrow{AD}$ (2)

where D divides $\overline{BC}$ internally by the ratio n : m

b) $m\ \overrightarrow{AB} - n\ \overrightarrow{AC} = (m - n)\ \overrightarrow{AE}$ assuming $m > n$ (3)

where E divide $\overline{CB}$ externally by the ratio m : n from B.

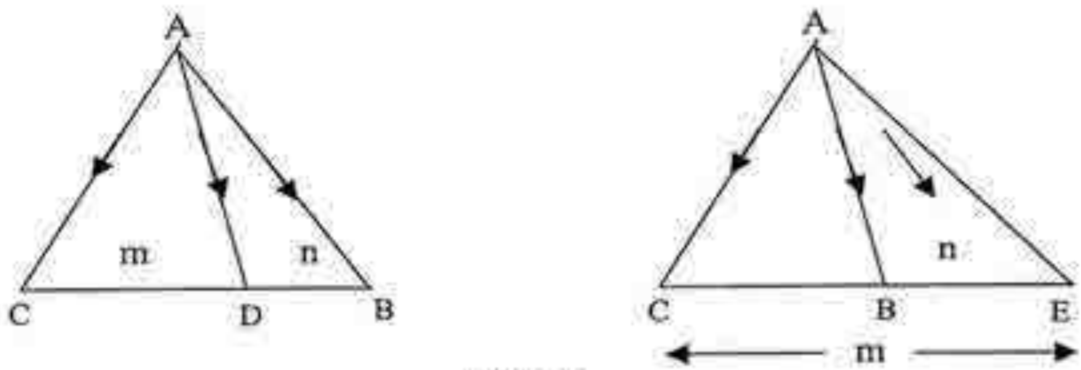


Fig. (49)

Proof : (Proof is not required)

a) Since D divides $\overline{BC}$ internally in the ratio n : m.

$\therefore \quad \frac{DB}{DC} = \frac{n}{m} \qquad\qquad \therefore \quad m\,(DB) = n\,(DC)$

From this relation and from fig. (49), noticing the position of the point D on $\overline{CB}$ we can write the folloiwng vector equality :

$m\,\overrightarrow{DB} = -n\,\overrightarrow{DC} \qquad \therefore \quad m\,\overrightarrow{DB} + n\,\overrightarrow{DC} = \overrightarrow{O}$

Using the rule of the triangle of vectors, we have

$$m\,\overrightarrow{AB} + n\,\overrightarrow{AC} = m\,(\overrightarrow{AD} + \overrightarrow{DB}) + n(\overrightarrow{AD} + \overrightarrow{DC})$$
$$= (m+n)\,\overrightarrow{AD} + (m\,\overrightarrow{DB} + n\,\overrightarrow{DC})$$
$$= (m+n)\,\overrightarrow{AD} + \overrightarrow{O} = (m+n)\,\overrightarrow{AD}$$

Result : If D is the midpoint $\overline{CB}$ then $\overrightarrow{AC} + \overrightarrow{AB} = 2\,\overrightarrow{AD}$

b) Since E divides $\overline{CB}$ externally in the ratio m : n

$\therefore \quad \frac{EB}{EC} = \frac{n}{m} \qquad\qquad \therefore \quad m\,(EB) = n\,(EC)$ (notice that m>n)

From this relation and from fig. (49), noticing the position of E on $\overline{CB}$ we can write the following vector equality :

$m\,\overrightarrow{EB} = n\,\overrightarrow{EC} \qquad \therefore \quad m\,\overrightarrow{EB} - n\,\overrightarrow{EC} = \overrightarrow{O}$

Using the rule of the triangle of vectors, we have :

$$m\,\overrightarrow{AB} - n\,\overrightarrow{AC} = m\,(\overrightarrow{AE} + \overrightarrow{EB}) - n\,(\overrightarrow{AE} + \overrightarrow{EC})$$
$$= (m-n)\,\overrightarrow{AE} + (m\,\overrightarrow{EB} - n\,\overrightarrow{EC})$$
$$= (m-n)\,\overrightarrow{AE} + \overrightarrow{O} = (m-n)\,\overrightarrow{AE}$$

and this proves relation (3)

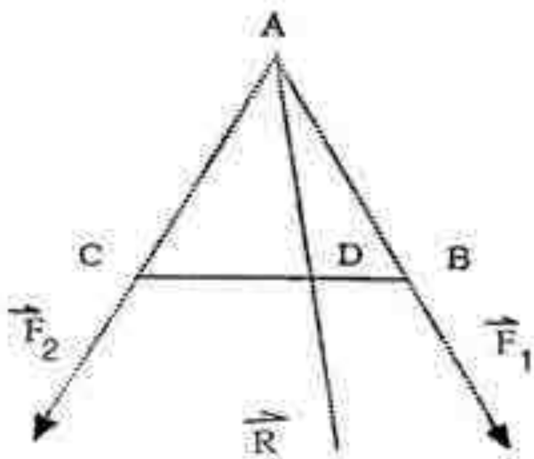
Using the above theorem we can find the line of action of several parallel foces, but will consider a system of two forces only.

Resultant of two parallel forces having the same direction* :

Consider two forces $\overrightarrow{F_1}$, $\overrightarrow{F_2}$ meeting in a point, say A, the angle between them is an acute angle as in fig (50-a). Let F_1, F_2 be their magnitudes respectively.

Let $\frac{F_1}{F_2} = \frac{m}{n}$

Take two points B, C on the lines of action of the forces $\overrightarrow{F_1}$, $\overrightarrow{F_2}$ so that AB = AC. Choose a suitable drawing scale for the magnitude of the force so that the force $\overrightarrow{F_1}$ is completely represented by the directed straight segment $m\,\overrightarrow{AB}$ thus the force $\overrightarrow{F_2}$ will be completely represented by the directed straight segment $n\,\overrightarrow{AC}$.



$\overrightarrow{R} = \overrightarrow{F_1} + \overrightarrow{F_2}$

Fig. (50-a)

The resultant of the two forces

$\overrightarrow{R} = \overrightarrow{F_1} + \overrightarrow{F_2}$ will be completely represented by the sum

$(m\,\overrightarrow{AB} + n\,\overrightarrow{AC})$ and this is equal to $(m + n)\,\overrightarrow{AD}$ where DE $\overrightarrow{BC}$ and divides it by the ratio n : m.

i.e. $\frac{DB}{DC} = \frac{n}{m} = \frac{F_2}{F_1}$ $\qquad \therefore\ DB \times F_1 = DC \times F_2.$ $\qquad$ (4)

* **Proof is not required.**

Now if we left the point A to move far away from $\overline{BC}$ without limit so that the triangle ABC remains an isosceles triangle. The two forces $\overrightarrow{F_1}$, $\overrightarrow{F_2}$ tend to be two parallel forces, having the same direction as in fig. (50-b) while the point D remains in its position on $\overline{BC}$ since the ratio m : n depends only on the ratio between the magnitudes of the two forces, and it is independent of their directions.

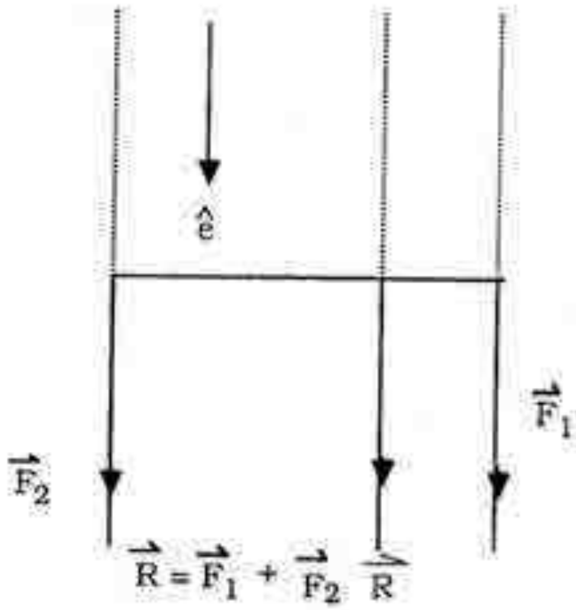


Fig. (50-b)

Let $\hat{e}$ be a unit vector in the direction of the two forces as in fig (50-b)

$\therefore \quad \overrightarrow{F_1} = F_2\,\hat{e},\ \overrightarrow{F_2} = F_2\,\hat{e}$

$\therefore \quad \overrightarrow{R} = \overrightarrow{F_1} + \overrightarrow{F_2} = (F_1 + F_2)\,\hat{e}$

This means that the resultant has the same direction as the two forces, and its magnitude is equal to the sum of magnitudes of the two forces. Summing up our results we have the following rule :

Rule :

The resultant of two parallel forces having the same direction, is a force in the same direction as the two forces, its magnitude is equal to the sum of the mangnitudes of the two forces, and its line of action divides the distance between the lines of action of the two forces in an inverse ratio to their mangitudes.

Resultant of two parallel forces, having opposite directions* :

Consider two forces $\overrightarrow{F_1}$, $\overrightarrow{F_2}$ meeting at A, the angle between them is an obtuse angle as in fig. (50-c). Let F_1, F_2 be their magnitudes respectively. We are going to assume that :

$$\frac{F_1}{F_2} = \frac{m}{n} \quad (m > n)$$

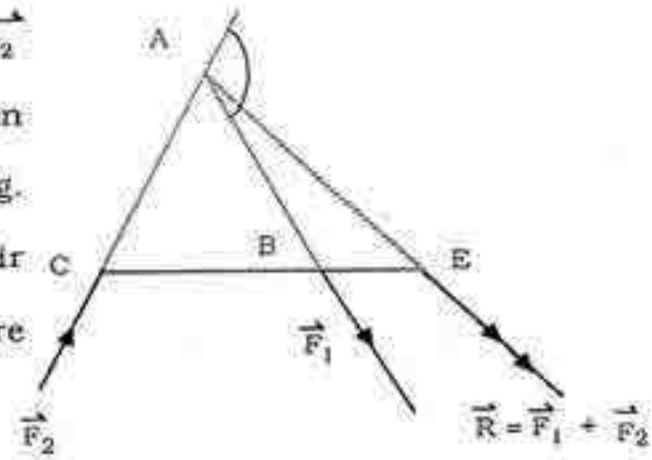


Fig. (50- c)

This means that the magnitude of $\overrightarrow{F_1}$ is greater than the magnitude of $\overrightarrow{F_2}$

Take two points B, C on the lines of action of $\overrightarrow{F_1}$, $\overrightarrow{F_2}$ so that AB=AC.Choose a suitable drawing scale to the magnitude of the force, so that the force $\overrightarrow{F_1}$ is completely represented by the directed

- **Proof is not required**

straight segment m $\overrightarrow{AB}$, then the force $\overrightarrow{F_2}$ will be completely represented by the directed straight segment (-n $\overrightarrow{AC}$) since the direction of the force $\overrightarrow{F_2}$ is opposite to that of $\overrightarrow{AC}$.

The resultant $\vec{R} = \overrightarrow{F_1} + \overrightarrow{F_2}$, will be completely represented by the sum (m $\overrightarrow{AB}$ -n $\overrightarrow{AC}$) and this is equal to (m-n) $\overrightarrow{AE}$, where E is a point on $\overline{CB}$ and divides $\overline{CB}$ externally in the ratio m : n from B.

i.e. $\dfrac{EB}{EC} = \dfrac{n}{m} = \dfrac{F_2}{F_1}$

$\therefore \quad EB \times F_1 = EC \times F_2$

Now is we let point A move far away from $\overline{BC}$ without limit so that the triangle ABC remains an isosceles triangle. The two forces $\overrightarrow{F_1}$, $\overrightarrow{F_2}$ tend to be two parallel forces, having opposite directions (without the vanishing of their resultant) as in fig.(50-d) while point E remains in its position on $\overrightarrow{CB}$ since the ratio m : n depends only on the ratio between the magnitudes of the two forces, but not on their directions.

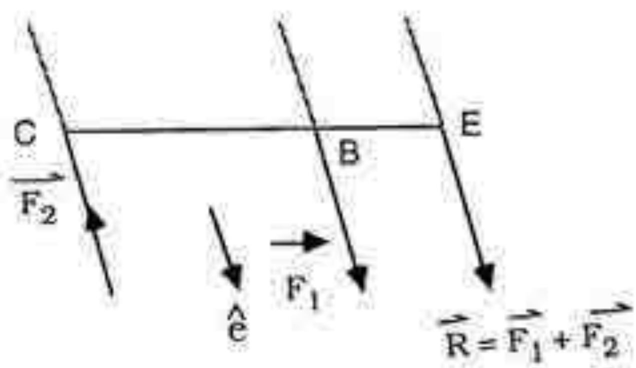


Fig. (50-d)

Let $\hat{e}$ be a unit vector in the direction of the force of greater magnitude.

$\overrightarrow{F_1} = \overrightarrow{F_1}\ \hat{e}$ $\qquad\qquad$ $\overrightarrow{F_2} = -F_2\ \hat{e}$

$\overrightarrow{R} = \overrightarrow{F_1} + F_2 = (F_1 - F_2)\ \hat{e}$

This means that the resultant has the same direction as the force of greater magnitude, and its magnitude is equal to the difference between the magnitudes of the two forces.

Summing up our results we have the following rule :

Rule :

The resultant of two parallel forces having opposite direction , is a force in the direction of the force of greater magnitude (the two forces are to of equal magnitude), its magnitude is equal to the difference between their magnitudes, and its line of action divides the distance between the lines of action of the two forces externally from the side of the force of greater magnitude in an inverse ratio to their magnitudes.

Moments of parallel coplanar forces:

Theorem

The sum of the moments of a finite number of parallel coplanar force about any point in its plane is equal to the moment of the resultant of these forces about the same point.

Proof :

We start to prove this theroem in the special case when the system is composed of two forces only.

1- If the two forces have the same direction :

Consider any point such as O in the plane of the two forces. From O draw the common perpendicular to the lines of action of the two forces $\overrightarrow{F_1}$, $\overrightarrow{F_2}$ to meet them at B, C respectively and to meet the line of action of the resultant at the point D. From equation (4) $DB \times F_1 = DC \times F_2$.

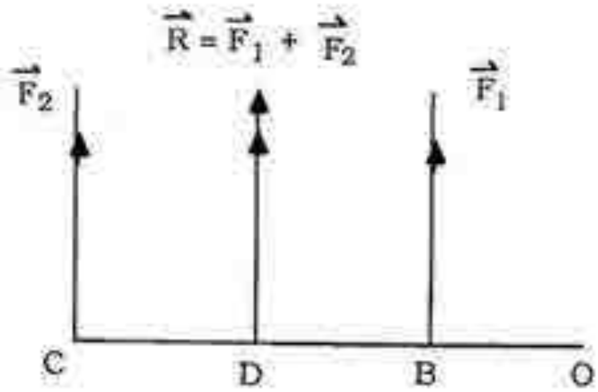


Fig. (51-a)

Since the forces are coplanar and O lies in its plane, we can use the algebraic measure of the moment referred to a unit vector $\hat{c}$ perpendicular to the plane of the force instead of the moment vector as previously explained. Moment of the system about O

$= -F_1 \times OB - F_2 \times OC.$

$$= -F_1 \times (OD - DB) - F_2 \times (OD + DC)$$

$$= -(F_1 + F_2) \times OD + (F_1 \times DB - F_2 \times DC)$$

$$= -(F_1 + F_2) \times OD$$

= Moment of the resultant about O.

2- If the two forces have opposite directions :

Consider any point such as O in the plane of the two forces. From O draw the common perpendicular to the lines of action of te two forces to meet them at B, C respectively and to meet the line of action of the resultant at the point E say. Assuming that $F_1 > F_2$. Then relation (5) gives $EB \times F_1 = EC \times F_2$. Moment of the system about O

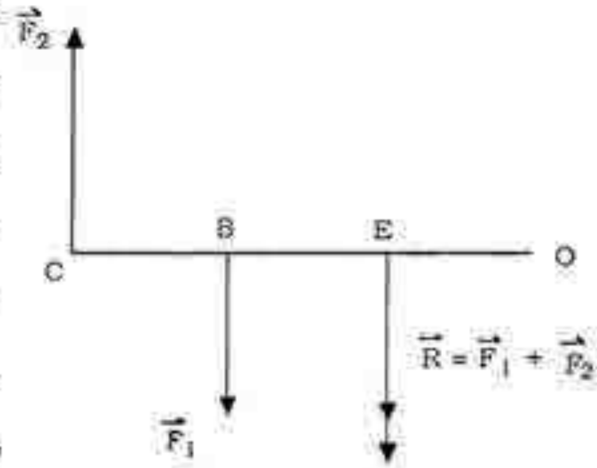


Fig. (51-b)

$= F_1 \times OB - F_2 \times OC$

$= F_1 \times (OE + EB) - F_2 \times (OE + EC)$

$= (F_1 - F_2) \times OE$

$=$ Moment of resultant about O.

3- If the system consists of a finite number of forces (more than 2) whose resultant does not vanish, the theorem can be proved by obtaining the resultant of any two forces whose resultant does not vanish and applying the theorem in pairs and so on until we obtain the resultant of the system.

4- The theorem is true if the caplanar forces are not parallel.

Example (1) :

Two parallel forces, whose magnitudes are 60 , 40 newtons, the distance between their lines of action is 50 cm. Find their resultant in the two cases.

a. The two forces are in the same direction.

b. The two forces are in opposite directions.

Solution :

a. Let $\hat{e}$ be a unit vector in the direction of the two forces, fig. (52-a)

$\therefore \overrightarrow{F_1} = 60\ \hat{e},\ \overrightarrow{F_2} = 40\ \hat{e}$

$\therefore \overrightarrow{R} = 100\ \hat{e}$, R = 100 newtons

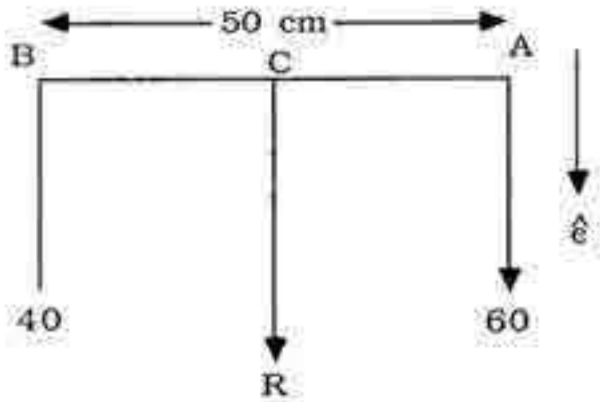


Fig. (52-a)

Draw the common perpendicular to meet their lines of action at A and B, respectively, let $C \in \overline{AB}$ be a point on the line of the resultant.

$\therefore 60 \times AC = 40 \times CB$

$\therefore 60 \times AC = 40 \times (50 - AC)$

$\therefore 100\ AC = 2000$

$\therefore AC = 20$ cm

b. Let $\hat{e}$ ba a unit vector in the directon of the force of greater magnitude fig. (52-b).

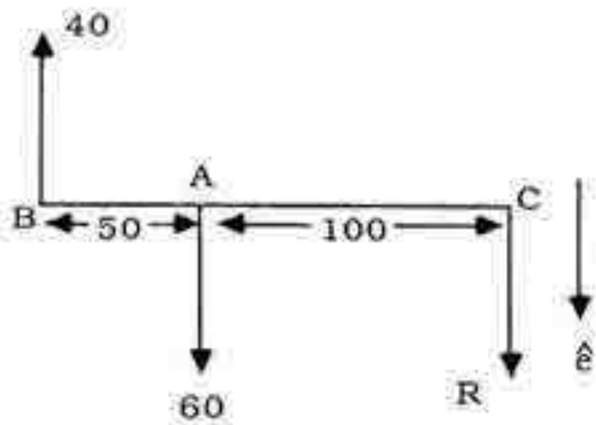


Fig. (52-b)

$\therefore \overrightarrow{F_1} = 60\ \hat{e},\ \overrightarrow{F_2} = 40\ \hat{e}$

$\therefore \overrightarrow{R} = 20\ \hat{e},\ R = 20$ newtons

The line of action of the resultant passes through a point C on the ray $\overrightarrow{BA}$, $C \notin \overline{BA}$, so that

$$60 \times AC = 40 \times CB$$

$\therefore 60 \times AC = 40 \times (50 + AC)$

$\therefore 20\ AC = 2000$

$\therefore AC = 100$ cm.

Example (2) :

Two parallel forces $\overrightarrow{F_1}$, $\overrightarrow{F_2}$, the magnitude of the first is 100 newtons, the magnitude of their resultant 150 newtons, the distance between the line of action of the first force and the resultant is 75 cm. Find the magnitude and direction of the second force $\overrightarrow{F_2}$ in the two cases :

a) $\overrightarrow{F_1}$ and $\overrightarrow{R}$ in the same direction.

b) $\overrightarrow{F_1}$ and $\overrightarrow{R}$ in opposite directions.

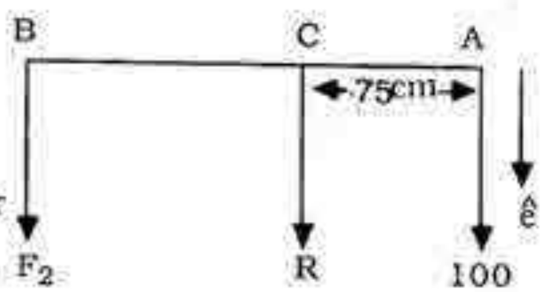


Fig. (53-a)

Solution :

a) Draw the common perpendicular to the line of action of $\overrightarrow{F_1}$ and $\overrightarrow{R}$ to meet them at A and C respectively, and let $\hat{e}$ be a unit vector in the direction of the resultant fig. (53-a).

$\therefore\ \overrightarrow{F_1} = 100\ \hat{e},\ \overrightarrow{R} = 150\ \hat{e}$

$\therefore\ \overrightarrow{F_2} = \overrightarrow{R} - \overrightarrow{F_1} = 50\ \hat{e}$

i.e. the force $\overrightarrow{F_2}$ is in the direction of the first and of magnitude 50 newtons. Its line of action passes through a point B on $\overrightarrow{AC}$, B $\notin$ $\overline{AC}$ so that

$100 \times AC = 50 \times CB$

$\therefore\ 100 \times 75 = 50 \times CB$

$\therefore\ CB = \frac{100 \times 75}{50} = 150$ cm.

b) If the resultant and the known force $\overrightarrow{F_1}$ are in opposite directions, let $\hat{e}$ be a unit vector in the direction of the resultant fig. (53-b).

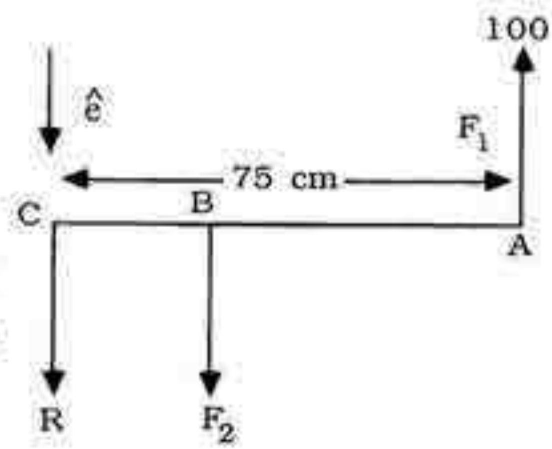


Fig. (53 - b)

$\therefore\ \overrightarrow{F_1} = -100\ \hat{e},\ \overrightarrow{R} = 150\ \hat{e}$

$\therefore\ \overrightarrow{F_2} = \overrightarrow{R} - \overrightarrow{F_1} = 250\ \hat{e}$

$\therefore$ The force $\overrightarrow{F_2}$ is in a direction opposite to that of $\overrightarrow{F_1}$ and of magnitude 250 newtons. Its line of action passes through a point B $\in$ $\overline{AC}$ so that

$100 \times 75 = 250 \times BC$ $\qquad \therefore\ BC = \frac{100 \times 75}{250} = 30$cm

Example (3) :

Find the resultant of four parallel forces, in the same direction and of magnitudes 3, 4, 1, 2, kg. wt acting at the point A, B, C, and D respectively which line on the same straight line perpendicular to the direction of the forces, given that AB = BC = 100cm, CD = 150cm.

Solution :

Let $\hat{e}$ be a unit vector in the direction of the forces, as in fig. (54).

$\vec{R} = 3\hat{e} + 4\hat{e} + 1\hat{e} + 2\hat{e} = 10\hat{e}$

$\therefore$ R = 10 kg. wt.

$\therefore$ The resultant is in the same direction as the forces, and of magnitde 10 kg. wt.

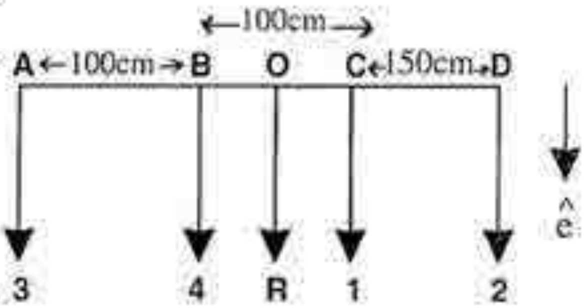


Fig. (54)

The line of action of the resultant passes through a point O determined as follows :

The algebraic measure of the moment of the resultant about A = The sum of the algebraic measures of the moments of the forces about A .

M = -4 × 100 - 1 x 200 - 2 × 350 = - 1300 kg.wt.cm

i.e. the resultant tends to rotate about A in a clockwise direction which means that the point O lies on the ray $\overrightarrow{AB}$.

$\therefore$ M = -R × AO

Comparing the last two relations

-R × AO = -1300 $\quad\quad \therefore AO = \dfrac{1300}{10} = 130$cm.

$\therefore$ O is at a distance of 130 from A.

Exercises (4)

1. $\overrightarrow{F_1}$, $\overrightarrow{F_2}$ are two parallel forces acting at the points A, B. F_1 = 40 newtons, F_2 = 70 newtons, AB = 50cm. Find the resultant of $\overrightarrow{F_1}$, $\overrightarrow{F_2}$ if :

   1st they the same direction

   2nd they have the opposite directions.

2. $\overrightarrow{F_1}$, $\overrightarrow{F_2}$ are two parallel forces having the same direction, the distance between their lines of action is 20cm. If the magnitude of their resultant is 50 N. and its line of action is distant 4cm from $\overrightarrow{F_1}$, find the magnitudes of the two forces.

3. The magnitude of the resultant of two parallel forces is 250 newtons while the magnitude of one of the forces is 150 newtons and acts at a distance of 40cm, from the resultant. Find the magnitude of the second force, and the distance between the line of action of the two forces if the known force and the resultant;

   i. have the same direction

   ii. have two opposite directions.

4. The magnitude of the resultant of two parallel forces is 350 newtons while the magnitude of one of the forces is 500 newtons and acts at a distance of 51cm from the resultant. Find

the second force and the distance between the lines of action of the two forces.if the known force and the resultant;

i. have the same direction.

ii. have opposite directions.

5. The magnitude of the smaller one of two parallel forces is 30 newtons acts at the end A of a light rod AB, while the larger one acts at the other end B. If the magnitude of their resultant is 10 newtons and its line of action is at a distance of 90 cm from the end B, what is the length of the rod?

6. Two parallel forces, in the same direction, of magnitudes F and 2 F act at the points A, B respectively. If the force $\vec{F}$ moves parallel to itself a distance x on the ray $\overrightarrow{BA}$, prove that the resultant of the two forces moves a distance $\frac{1}{3}$ x in the same direction.

7. Three equal and parallel forces, in the same direction act at the vertices A, B, C of a triangle ABC. Prove that their resultant passes through the point of intersection of the medians of the triangle.

8. Four equal and parallel forces, in the same direction act at the vertices A, B, C, D, of a square ABCD. Prove that their resultant passes through the point of intersection of its diagonals.

9. The points A, B, C, D, E, lie on the same straight line so that AB

= 4cm, BC = 6cm, CD = 8cm, DE = 10cm. Five forces of magnitudes 60, 30, 50, 80, 40, kg. wt act at the points A, B, C, D, E, respectively and in a direction perpendiculat to $\overleftrightarrow{AE}$, so that the first three forces are in the same direction and the last two forces in the opposite direction. Find the resultant of the system of forces.

10. A, B, C, D, are four points on a straight line such that AB = 32 cm, BC = 40 cm, CD = 8 cm. Four forces 8, 10, 7, 3, newtons act at the points A, C, B, D respectively so that the first two forces are in the same direction Find the resultant of the system of these forces and the distance between its point of intersection with $\overline{AD}$ and point A.

11. Five parallel like forces 4, 6, 2, 8, 10, newtons act at A, B, C, D, E, which are on a straight line perpendicular to the direction of forces. Find the point of application of resultant if AB = CD = 60cm, BC = 2 DE = 90cm.

12. A, B, C, D, H are five points on one straight line where 5 AB = 3 BC = CD = 6 DH = 30cm. Four parallel forces 8, 12, 16, F newtons act at A, C, H, D, respectively in a directio perpendicular to the straight line $\overleftrightarrow{AH}$, so that the three forces are in the same direction and the force F in the opposite direction. Find F if the resultant of the system of forces pases through B.

EQUILIBRIUM OF FORCES

In this article we are going to study the equilibrium of a rigid body under the action of a number (more than three) of coplanar parallel forces. In this case we say that the system of forces is in equilibrium.

Experiment :

Equilibrium of a rigid body under the action of many parallel forces.

The aim of the experiment is to verify that if a rigid body is in equilibrium under the action of a system of parallel coplanar forces (whose number is more than three), then :

1. The sum of the algebraic measures of these forces is zero.
2. The sum of the algebraic measures of the moments of these forces about an point in its plane is zero.

Apparatus :

Consists of a light graduated ruler, two capstan holders. Two spring balances, weights and strings.

1. Fix the two springs, each in a holder then suspend the reuler by means of strings until the balances and strings become vertical as in fig. (55).
2. Suspend a number of suitable weights in the ruler by means of the strings and adjust the positions and magnitudes of the

weights until the ruler is in equilibrium in a horizontal position. Let the magnitudes of the weights be W_1 , W_2 , W_3, ..., W_n (the direction of these forces is vertically downwards).

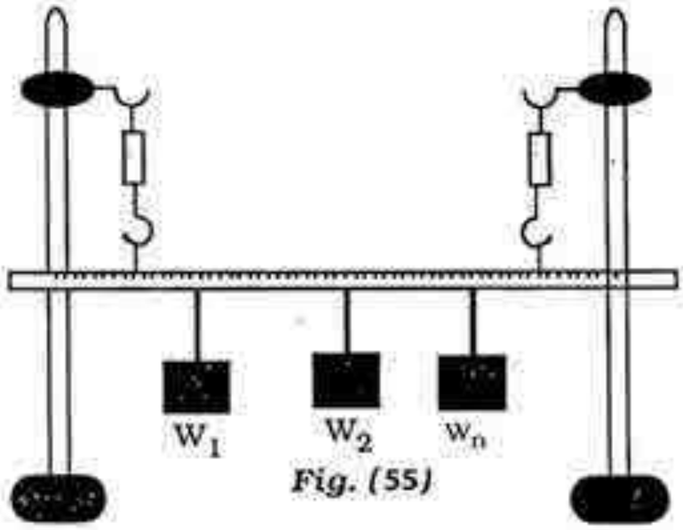


Fig. (55)

3. Record the readings of the balances to find the tensions, let their magnitudes be T_1 T_2 (These two forces act vertical upwards, we find that $T_1 + T_2 = W_1 + W_2 ...... + W_n$.
4. Measure the distances of the points of supension of the weights from a point on the ruler (say the midpoint of the ruler).
5. Calculate the algebraic sum of the moments of all forces : W_1, W_2, ... , W_n , T_1 , T_2 acting on the ruler about the chosen point, we find that it is equal to zero.
6. Repeat the experiment several times, changing the weights and their points of suspension.

In every case we get the following result :

If a body is in equilibrium under the action of a system of parallel coplanar forces, then :

1. the sum of the algebraic measures of these forces is zero.
2. the sum of the algebraic measures of the moments of these forces about any point in its plane is zero.

From this experiment we can formulate the folloiwng rule :

Rule :

If a rigid body is in equilibrium under the action of a system of parallel coplanar forces, then :

1. the sum of the algebraic measures to these forces (with respect to a parallel unit vector) is equal to zero.
2. the sum of the algebraic measures of the moments of these forces about any point in its plane is equal to zero.

Example (1) :

A uniform rod of length 1m and whose weight is 50 N. (acting at its midpoint) is suspended from its ends by two, vertical strings and carries two weights, one of which is 15 newtons, from a point distant 20cm from one end and the other 20 newtons at a point 30cm distant from the other end. Find the tension in each string.

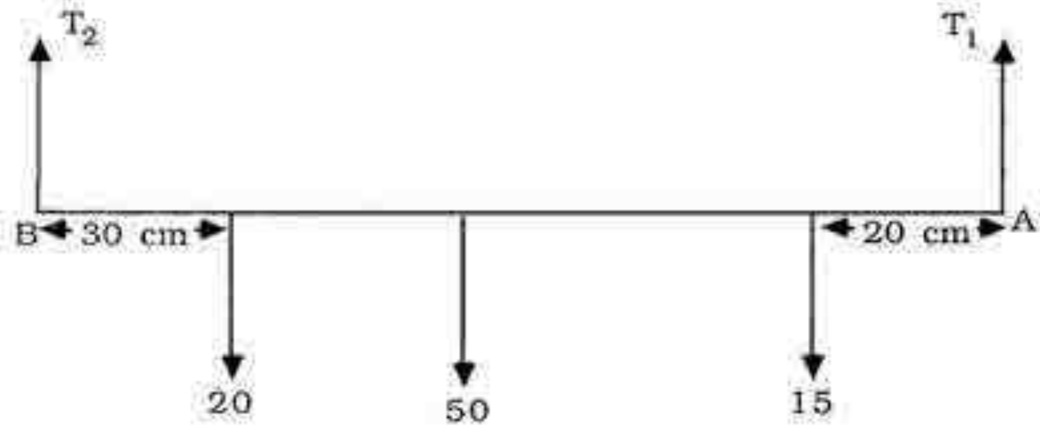


Fig. (56)

Solution :

The rod is in equilibrium under the action of five forces which are, the tension $\overrightarrow{T_1}$ at the end A, the tension $\overrightarrow{T_2}$ at the end B, the weight of the rod act at its midpoint, the two weights 15 and 20 newtons.

$\therefore$ The sum of the algebraic measures of the forces is zero.

$\therefore$ $T_1 + T_2 - 15 - 50 - 20 = 0.$

$\therefore$ $T_1 + T_2 = 85$ newtons. (1)

By taking moments about A, the sum of the algebraic measures of the moments is equal to zero.

$\therefore$ $15 \times 20 + 50 \times 50 + 20 \times 70 - T_2 \times 100 = 0.$

i.e. $300 + 2500 + 1400 = 100\ T_2.$

$\therefore$ $T_2 = 42$ newtons.

From (1) $T_1 = 85 - T_2$

$= 43$ newtons

Example (2) :

AB is a uniform rod of length 90cm and whose weight is 60 newtons (acting at its midpoint) is suspended in a horizontal position b two vertical strings at its ends A, B, where a weight of 150 newtons may be suspended in order that the tension at A be twice the tension at B.

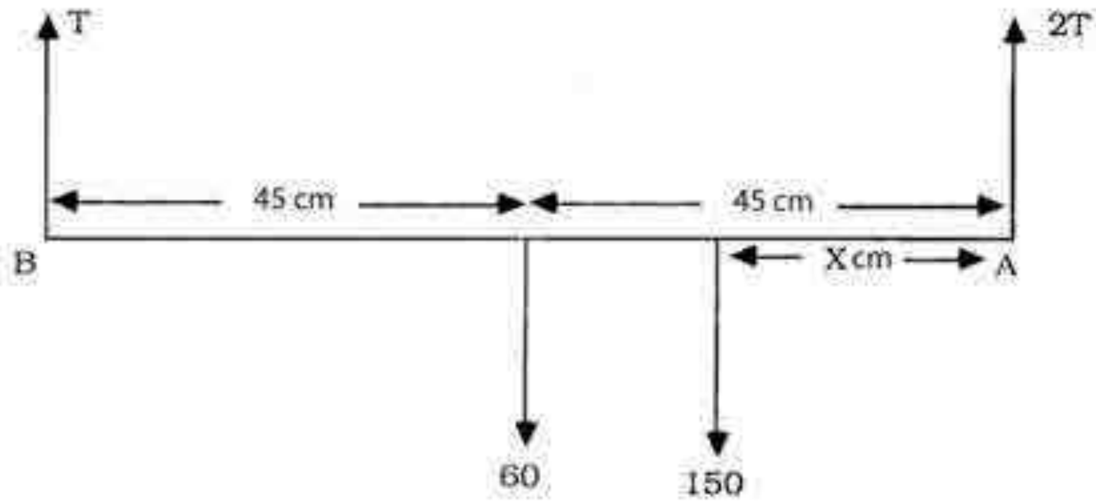


Fig. (57)

Let the 150 N. weight be suspended from a point distant x cm from A, let the tension at B be T, i.e. the tension at A is equal to 2 T.

The sum of algebraic measure of the forces = 0.

$\therefore\ 2T + T - 150 - 60 = 0.$

$\therefore\ 3T = 210 \qquad \therefore\ T = 70$ newtons.

Taking moments about A

$150 \times x + 60 \times 45 - T \times 90 = 0.$

$\therefore\ 150x + 2700 - 6300 = 0.$

$\therefore\ 150x = 3600 \qquad \therefore\ x = 24$ cm

Example (3) :

$\overline{AB}$ is a rod of length 150 cm and weight 140 N (acting at its midpoint). The rod rests in a horizontal position on two supports one of them at A and the other at a point C 25 cm distant from B. Find the pressure on each support, find also the least weight that can be suspended at B in order that the pressure of the rod on the support at A vanishes. What is the value of the pressure on the support at C at this instant ?

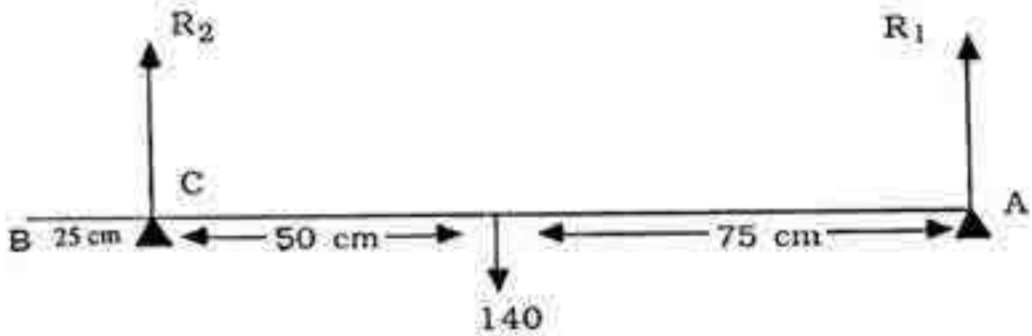


Fig. (58)

Let R_1 . R_2 be the reactions of the supports at A and C respectivel fig. (58) Conditions of equilibrium : The sum of the algebraic measures of the forces = 0.

$\therefore R_1 + R_2 = 140$ newtons.

The sum of the algebraic measures of the moments of the forces about C = 0.

$\therefore R_1 \times 125 - 140 \times 50 = 0.$

$\therefore\ R_1 = 56$ newtons

= pressue at A

and $R_2 = 140 - R_1 = 84$ newtons.

= pressure at C.

Now let us suspend a weight W from the end B so that the pressure of the rod on the support at A vanishes. At this instant the end A is about to leave the support, and thus we put $R_1 = 0$.

Referring to fig. (59) and equating the sum of the algebraic measures of the moments of the forces about C to zero, we find that

$-140 \times 50 + W \times 25 = 0$

$\therefore\ W = 280$ newtons.

Let R be the reaction of the support at C on the rod.

$\therefore\ R = 140 + W$

$= 140 + 280 = 420$ newtons

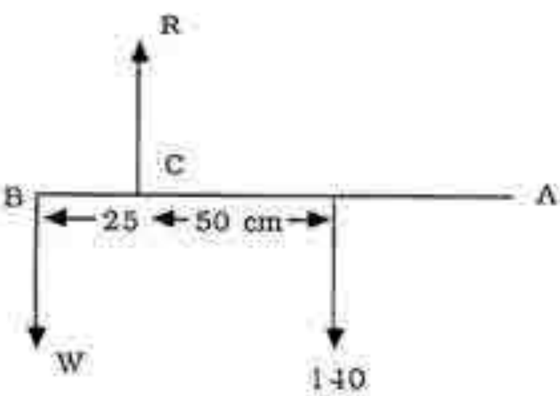


Fig. (59)

Exercises (5)

1. An iron beam of length 30cm whose weight is 20 newtons (acting at its midpoint) rests in a horizontal position on two supports, one of them at one end and the other at a distance of 10cm from the other end. Find the reaction of each of the two supports on the beam.
2. A beam of negligible weight and of length 120cm rests in a horizontal position on two supports at its two ends. At what point of the beam a weight of 12 kg. wt should be suspended so that the reaction at one end may be twice the reaction at the other end ?
3. An iron beam of length 50cm whose weight is 75 newtons (acting at its midpoint) rests in a horizontal position on two supports, the distance between them is 24cm. If the pressure on one of the supports is twice the other, find the distance between each support and the end of the beam near to it.
4. A rod of length 120cm and of negligible weight is suspended in a horizontal position by means of two vertical strings at its ends. Two weights of magnitudes 5 and 8 newtoons are suspended at its points of trisections. find the tension in each string.
5. A rod of length 90cm, and of negligible weight rests in a

horizontal positon on two supports at its points of trisections. Two weights of magnitudes 20 and 30 newtons are suspended from its ends. Find the pressure on each support.

6. AB is a ruler of length 50 cm and of weight 500 gm.wt (acting at its midpoint) is suspended in a horizontal position by means of two vertical strings at its ends weights are suspended one of weight 1.5 kg. wt at a point 10cm distant from A, the other of weight 2 kg. wt at a point 15cm distant from B. Find the tension in each string.

7. AB is a uniform rod of length 80cm and weight 4 kg.wt acting at its midpoint rests in a horizontal position on two supports, one of them is at a point 10cm distant from A, and the second at a distance of 20cm from B. Two weights of magnitudes 3 and 5 kg. wt are suspended from the rod at points 20cm distant from A, 30 cm from B respectively. Find the pressure on each support.

8. AB is a ruler of length 90cm and weight 6 newtons acting at its midpoint, is suspended in a horizontal position by means of two vertical strings at its ends. Where should a weight of magnitude 15 newtons be suspended in order that the tension in one of the strings is twice that in the other string ?

9. AB is a rod of length 100 cm and weight 10 newtons acting at its midpoint rests in a horizontal position on two supports one of

them at A and the other at a point 25 cm distant from B. What is the magnitude of the weight that should be suspended from the end B of the rod so that the reaction on the support nearer to this end will be six times its value at A, what is the value of the reactions then ?

10. AB is a rod of length 80 cm and weight 35 newtons, acting at its midpoint, rests in a horizontal position on two supports at its ends, and carries a weight 5 newtons at a point 20 cm distant from B. At what point of the rod should a weight of magnitude 20 newtons be suspended in order that the reaction at B be twice its value at A? What are the magnitudes of the reactions then ?

11. AB is a rod of length 50cm and weight 10 newtons, acting at its midpoint, rests in a horizontal position on two supports one of them at a point 15 cm from A, and the other at a point 10 cm distant from B. Find the pressure on each support. What is the magnitude of the weight that should be suspended from the end B, so that the rod is on the point of rotation, and what is the value of the pressure on the support at this instant ?

12. AB is a rod of length 90 cm, and weight 50 newtons, acting at its midpoint, rests in a horizontal position on two supports, one of them at the end A and the other at a point 15 cm from B. Find the magnitude of the pressure on each support. Find also the

magnitude of the weight that should be suspended from the end B so that the rod is about to rotate. What is the magnitude of the pressure on the support at this instant ?

13. AB is a rod of length 120 cm and of weight 60 newtons, acting at its midpoint, rests in a horizontal position on a support at B, and is kept in equilibrium by means of a string attached to a point of the rod 40 cm distant from the endl A, and carries a weight of magnitude 20 newtons at a point 20 cm from A. Find the tension in the string and the pressure on the support. What is the magnitude of the weight that should be suspended from A in order that the rod is about to separate from the support, and what is the magnitude of the tension in the string at this instant ?

14. AB is a uniform rod of length 120 cm and whose weight is 600 gm.wt. The rod rests in a horizontal position on two supports C,D the distance between them is 60 cm such that AC = 25cm. A weight is hanged at H such that AH = 30cm. Find :

1^{st} the magnitude of each of the reactions at C , D if the weight at H = 200 gm.wt.

2^{nd} the weight at H if the reaction at C is twice the reaction at D.

15. A heavy uniform rod of length 140 cm is suspended by two vertical strings one of them at B, the other is 40 cm from A. If

the tension in the string at B equal 1/4 the tension in the other string, find where act the weight of the rod and if the greatest weight suspended from A equals 12 newtons, find the weight of the rod.

16. AB is a nonuniform rod of length 120cm rests in a horizontal position on a support at a point 30 cm from A carrying two weights of magnitudes 1 newton at B and 16 newtons at A. If the weight at A becomes 8 newtons the support will be at a point 40cm from A, find the weight of the rod and its point of application.

17. AB is a rod of length 1m and of weight 700gm.wt. (acts at its mid-point) rests on a support at B. The rod is kept horizontally in equilibrium by means of a string attached to a point of the rod 30cm distant from A. The rod carries a weight of magnitude 350gm.wt. suspended at a point 10cm from A. Find the tension in the string and the pressure on the support. What is the magnitude of the weight that should be suspended from A in order that the rod is about to separate from the support, and what is the magnitude of the tension in the string at this instant.

Chapter Four

General Equilibrium

Preface :

In this chapter we will deal with equilibrium of body under the action of set of forces.

Objectives:

By the end of this chapter, the student should be able to:

(1) Recognize when a set of coplanar forces are in equilibrium.

(2) Recognize the sufficient and necessary conditions of the equilibrium of a set of coplanar forces.

(3) Recognize the direction of the reaction of a hinge connected to one of the ends of a rod.

(4) Solve problems including equilibrium of a rod or a ladder rests on a horizontal ground and a vertical well.

(5) Recognize the magnitude of the least horizontal force that can be applied at the lower end of a rod rests on a rough horizontal ground that make the rod about to move towards the wall or away of it.

Subjects:

(1) Equilibrium of a body under a set of coplanar forces.

(2) The necessary and sufficient conditions of equilibrium of a set of coplanar forces.

(3) Equilibrium of rod or ladder on a rough horizontal ground and a smooth wall or a rough horizontal ground and a rough vertical wall.

Definition :

A set of coplanar forces is said to be in " equilibrium " if both the vector sum of the forces and the vector sum of the moments of the forces about any point vanish. Also when a body is acted on by such a set of coplanar forces, the body is said to be in equilibrium.

Theorem :

If the vector sum of a set of forces and the vector moment of forces about one point vanish, the set of force is in equilibrium.

Proof :

Suppose that the vector sum of the moments of the force about a point (o) vanishes.

Since the vector sum of the set of forces vanishes (i . e $\vec{R} = \vec{0}$), it follows from a previous theorem that the vector sum of the moments of the forces about a point does not alter from one point to another. Now if the vector sum of the moments of the forces about one point (o) vanishes, so it does, about any point. Therefore, the set is in equilibrium.

According to this theorem, the conditions under which a set of forces is in equilibrium may by formulated as follows

The necessary and sufficient conditions for the equilibrium of a set of coplanar forces:

For a set of coplanar forces to be in equilibrium it is necessary and sufficient that

1. the vector sum of the forces vanishes,
2. the vector sum of the moments of the forces about one point vanishes.

These conditions for the equilibrium of a set of coplanar forces may be expressed in a form which is more suitable for application.

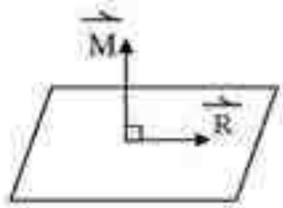


Figure (60)

Bearing in mind that we are only dealing with a set of coplanar forces and the points about which the moments are refered to lie in the plane of the forces, we conclude that :

- The vector sum of the forces $\vec{R}$ must lie in the plane of the forces.
- and the vector sum of the moments of the forces about any point in the plane of the forces, $\vec{M}$, must be normal to this plane, as shown in Fig. (60).

Now introduce the right-handed set of unit vectors $\vec{i}$, $\vec{j}$, $\vec{k}$ such that $\vec{i}$ and $\vec{j}$ lie in the plane of the forces. Thus $\vec{K}$ is along the normal to this plane.

It is now obvious that $\vec{M}$ is parallel to $\vec{K}$, while $\vec{R}$ may be resolved into components along $\vec{i}$ and $\vec{j}$ see Fig. (61).

$$\therefore \vec{R} = X\ \vec{i} + Y\ \vec{j}$$

$$\vec{M} = M\ \vec{k}$$

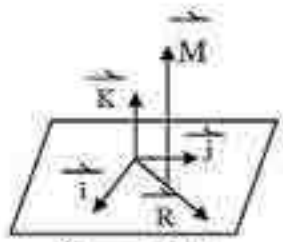


Figure (61)

Where :

$X =$ the algebraic sum of the components of the forces in direction of $\vec{i}$,

$Y =$ the algebraic sum of the components of the forces in direction of $\vec{j}$,

and

$M =$ the algebraic sum of the moments of the forces relative to $\vec{k}$.

We notice that if :

$$X = Y = M = 0 ,$$

then

$$\vec{R} = \vec{0} \quad \text{and} \quad \vec{M} = \vec{0}$$

Since the directions of $\vec{i}$ and $\vec{j}$ may be chosen in many different ways in the plane of the forces, then we are lead to the following equivalent formulation for the conditions of equilibrium of a set of coplanar forces:

For a set of coplanar forces to be in equilibrium, it is necessary and sufficient that :

1. the algebraic sum of the components of the forces in any two orthogonal directions in the plane of the forces must be zero.
2. The algebraic sum of the moments of all the forces about one point in the plane of forces must be zero.

In mathematical forms, these conditions are :

$$\boxed{\begin{array}{l} X = 0 \ , \\ Y = 0 \ , \\ M = 0 \end{array}}$$

Note :

The necessary and sufficient conditions are still true when the two unit vectors $\overrightarrow{i}$ and $\overrightarrow{j}$ are not parallel (and not necessarily perpendicular) .

Example (1) :

A uniform rod of weight 2 kg.wt. and is 100 cm. long. One of its ends is attached to a hinge fixed in a vertical wall. A weight of 2 kg.wt. is suspended at a point of the rod 75 cm. from the hinge. The rod is kept horizontally by means of a string attached in the other end and a point on the wall vertically above the hinge. If the string is inclined to the horizontal by an angle 60°, find the tension and the reaction of the hinge.

Solution :

Fig. (62) shows the equilibrium of the rod in a horizontal position under the following four forces :

1. the 2 kg. wt. of the rod directed vertically downwards and because of uniformity it acts at the midpoint of the rod.
2. The 2 kg. wt. acting vertically downwards at a point on the rod 75 cm. from the hinge.

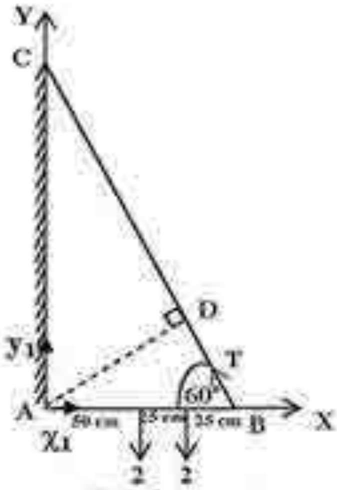


Figure (62)

3. The tension, T, in the string which is acting at the end B and having a line of action inclined at 60° to the horizontal.
4. The reaction at the hinge, which is acting at the end A which have two components x_1, y_1.

For the resolution of the forces we choose the two orthogonal directions $\overrightarrow{Ax}$ and $\overrightarrow{Ay}$ as shown in Fig. (62), and suppose the components of the reaction in these directions are (x_1 , y_1). Now we write the conditions for the equilibrium of the rod.

- The vanishing of the algebraic sum of the forces in the direction of $\overrightarrow{Ax}$ i . e.

$$X_1 - T\cos 60^\circ = 0 \qquad (1)$$

- The vanishing of the algebraic sum of the forces in the direction of $\overrightarrow{Ay}$ i . e.

$$y_1 + T\sin 60^\circ - 2 - 2 = 0 \qquad (2)$$

- The vanishing of the algebraic sum of the moments of forces about a point, A , say , i . e.

$$T \times AD - 2 \times 50 - 2 \times 75 = 0 \qquad (3)$$

Where A D is the length of the perpendicular from A on $\overline{BC}$

$$AD = AB \sin 60^\circ = 100 \times \frac{\sqrt{3}}{2} = 50\sqrt{3} \text{ cm}$$

Substituting in (3) we get.

$$T \times 50\sqrt{3} - 2 \times 50 - 2 \times 75 = 0$$

From which

$$T = \frac{5}{\sqrt{3}} \text{ kg. wt.}$$

Using this value in (1) and (2) we have :

$$x_1 = T \cos 60^\circ$$

$$= \frac{5}{\sqrt{3}} \times \frac{1}{2} = \frac{5}{2\sqrt{3}} \text{ kg. wt.}$$

$$y_1 = 4 - T \sin 60^\circ$$

$$= 4 - \frac{5}{\sqrt{3}} \times \frac{\sqrt{3}}{2} = \frac{3}{2} \text{ kg. wt.}$$

to evaluate the magnitude and direction of the reaction at the hings, let R be the magnitude of the reaction whose line of action is inclined with an angle θ to $\overrightarrow{AX}$ see Fig.(63), then we have :

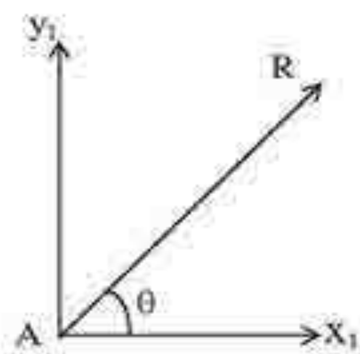


Figure (63)

$$R = \sqrt{x_1^2 + y_1^2} = \sqrt{\left(\frac{5}{2\sqrt{3}}\right)^2 + \left(\frac{3}{2}\right)^2}$$

$$= \frac{1}{2}\sqrt{\frac{52}{3}} = \sqrt{\frac{13}{3}} \text{ kg. wt.}$$

$$\tan\theta = \frac{y_1}{x_1} = \frac{\frac{3}{2}}{\frac{5}{2\sqrt{3}}} = \frac{3\sqrt{3}}{5}$$

$$\theta \simeq 46^\circ\ 7'$$

Example (2) :

A uniform ladder of weight 10 kg. wt. rests with its lower end A on a smooth horizontal ground and its upper end B against a smooth vertical wall. The ladder is being kept in a vertical plane and inclined at 45° to the ground by joining the lower end A by a string to the intersection point of the wall and the ground. If a man of weight 80 kg. ascends the ladder, find the tension of the string when the man has ascended three-quarters of the length of the ladder. Find also the maximum allowable tension given that the string was about to break when the man reached the top of the ladder.

Solution :

We notice that, because of uniformity, the weight of the ladder acts at its mid point.

Let L be the length of the ladder, R_1 be the ground reaction at A and R_2 be the wall reaction at B. Consider the

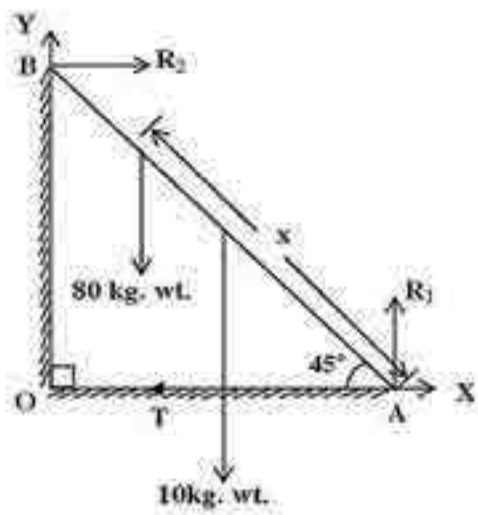


Figure (64)

vertical plane in which the ladder rests and take the orthogonal directions $\overrightarrow{Ox}$ and $\overrightarrow{Oy}$ in it, see Fig.(64), where O is a point on the ground vertically below B.

Suppose the man has ascended a distance x on the ladder. Resolving the forces in the $\overrightarrow{Ox}$ direction we have :

$$R_2 - T = 0$$

$$T = R_2 \qquad (1)$$

By taking moments about A we get

$$-R_2 \times L \times \frac{1}{\sqrt{2}} + 10 \times \frac{1}{2} L \times \frac{1}{\sqrt{2}} + 80 \times \frac{1}{\sqrt{2}} \times x = 0$$

$$R_2 = 5 + 80 \times \frac{x}{L} \qquad (2)$$

and on using (1), we have :

$$T = 5 + 80 \times \frac{x}{L}$$

From this result, we notice that, the tension increases with X i . e. with the distance ascended by the man on the ladder.

When the man ascends three quarters of the length of the ladder then:

$$\frac{x}{L} = \frac{3}{4}$$

Hence :

$$T = 5 + 80 \times \frac{3}{4} = 65 \text{ kg. wt.}$$

To find the maximum allowable tension in the string we consider the case where the man is at the top of the ladder i . e.

$$\frac{x}{L} = 1$$

For this value we have

$$T = 5 + 80 \times 1 = 85 \text{ kg. wt.}$$

Example (3) :

A uniform rod, of weight W, rests in a vertical plane with one end against a smooth wall and the other end on a rough horizontal floor such that the rod is in a vertical plane perpendicular to the floor. If the inclination of the rod to the floor is 45°, prove that the coefficient of friction between the rod and the floor cannot have a value less than $\frac{1}{2}$.

If the coefficient of friction is equal to $\frac{3}{4}$, find the horizontal force acting at the lower end of the rod so that the rod is about to move.

(i) towards the wall.

(ii) away from the wall.

Solution :

Let L be the length of the rod AB, R_1 be the reaction of the wall at the end A, R_2 be the normal reaction of the floor at the end B and F be the force of friction at B. Consider the vertical plane in which the rod rests and take the orthogonal directions $\overrightarrow{CX}$, $\overrightarrow{Cy}$

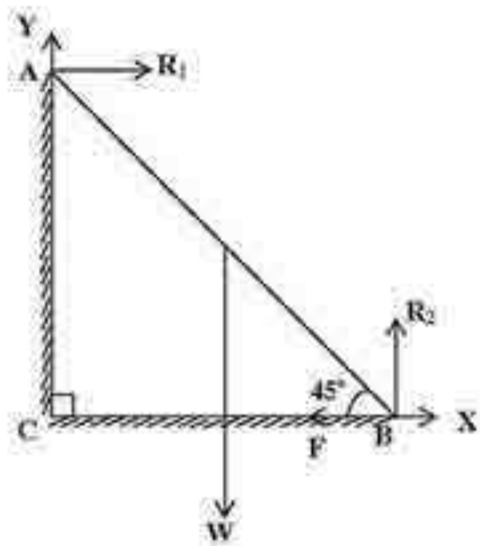


Figure (65)

as shown in Fig. (65), where C is a point on the floor directly below A. It is obvious that if the motion occures , the end B will move away from the wall and hence the force of friction is directed towards the wall.

By resolving the forces in the direction $\overrightarrow{CX}$ we get

$$R_1 - F = 0$$

$$F = R_1 \qquad (1)$$

By resolving the forces in the direction $\overrightarrow{CY}$, we get

$$R_2 - W = 0$$

$$R_2 = W \qquad (2)$$

Taking the moments about the end B we have

$$-R_1 \times \frac{L}{\sqrt{2}} + W \times \frac{L}{2\sqrt{2}} = 0$$

$$R_1 = \frac{W}{2} \qquad (3)$$

From (1) and (3) it follows that :

$$F = \frac{W}{2} \qquad (4)$$

Now let μ be the coefficient of friction, we know that

$$F \leqslant \mu R_2$$

Substitution of F and R_2 in this inequality gives '

$$\frac{W}{2} \leqslant \mu W$$

$$\mu \geqslant \frac{1}{2}$$

Let us now assume that $\mu = \frac{3}{4}$, and consider :

Case (i) : the rod is about to move towards the wall

The lower end of the rod is about to move towards the wall, let F_1 be the magnitude of the required applied force, which is directed towards the wall, see Fig. (66).

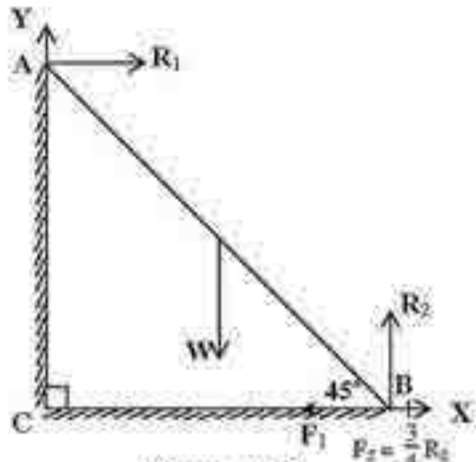


Figure (66)

Hence the force of friction is directed away from the wall and has a magnitude of $\frac{3}{4}R_2$ (a limiting friction)

Resolution of the forces along $\overrightarrow{CX}$ given :

$$F_2 - F_1 + R_1 = 0$$

$$\frac{3}{4}R_2 - F_1 + R_1 = 0 \qquad (5)$$

By resolving the forces along $\overrightarrow{CY}$: by taking the moments of the forces about B, we obtain equations (2) and (3) as above. Then using the values of R_1 and R_2 from (2) and (3) in equation (5) we have

$$\frac{3}{4}W - F_1 + \frac{W}{2} = 0$$

$$F_1 = \frac{5}{4}W$$

<u>Case (ii) :</u> the rod is about to move away of the wall

The lower end is about to slip away from the wall. Let F_2 be the required applied force, which is directed away from wall as in Fig. (67). Hence the force of friction is directed towards the wall and is equal to $\frac{3}{4}R_2$.

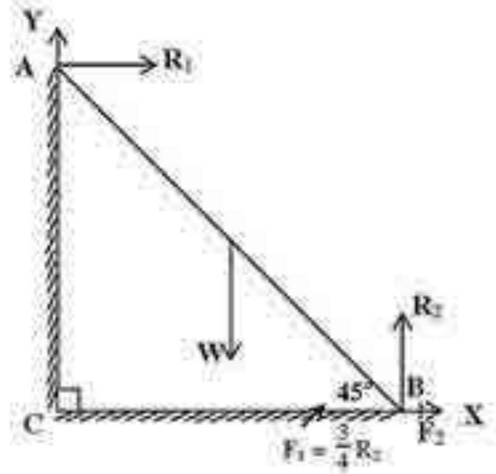


Figure (67)

Resolution of the forces in the direction $\overrightarrow{CX}$ gives

$$F_2 - F + R_1 = 0$$

$$F_2 - \frac{3}{4}R_2 + R_1 = 0$$

Equations (2) and (3) are still applied here so that :

$$F_2 - \frac{3}{4}W + \frac{W}{2} = 0 \quad \Rightarrow \quad F_2 = \frac{1}{4}W$$

Example (4) :

A uniform ladder is in equilibrium in a vertical plane resting against a vertical wall and a horizontal floor. If the measure of the angle of friction between the ladder and each of the wall the floor is λ, then prove that the measure of the angle of inclination of the ladder to the vertical when the friction is limiting is $\theta = 2\lambda$

Solution :

Let the weight of the ladder be W, L its length and θ is the angle measure with the vertical :

R_1 the normal reaction at A

R_2 the normal reaction at B

μ the coefficient of friction with both the wall and the floor. Since probable slide, is away of the wall then .

A is moving away of wall.

B moving down towards the floor.

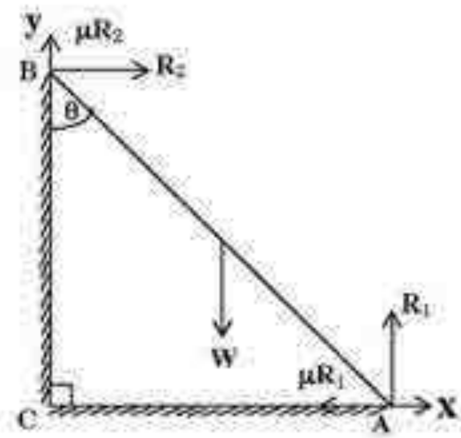


Figure (68)

Therefore : limiting friction at A is directed towards the wall C its value μR_1, limiting friction that at B is directed away of the floor and its value μR_2 Fig. (68))

In the vertical plane : take $\overrightarrow{CA}$, $\overrightarrow{CB}$ as axes, where C the interaction of the wall of the floor.

Resolving in direction $\overrightarrow{CA}$ we get :

$$R_2 - \mu R_1 = 0 \qquad (1)$$

Resolving in direction $\overrightarrow{CB}$ we get :

$$R_1 + \mu R_2 - W = o \qquad (2)$$

From (1) , (2)

$$\mu^2 R_1 + R_1 = W \qquad R_1 (1+\mu^2) = W \qquad (3)$$

$$R_1 = \frac{W}{1 + \mu^2}$$

$$\therefore R_2 = \frac{\mu W}{1 + \mu^2}$$

Take moment around B, we get :

$$\therefore R_1 \times L \sin\theta - \mu R_1 \times L \cos\theta - W \times \frac{L}{2} \sin\theta = 0$$

Dividing by $L \sin\theta$, we get :

$$R_1 (1 - \mu \cot\theta) = \frac{W}{2} \qquad (4)$$

Dividing (3) by (4), we get :

$$\frac{1 + \mu^2}{1 - \mu \cot\theta} = 2 \quad \Rightarrow \quad \cot\theta = \frac{1 - \mu^2}{2\mu}$$

But $\mu = \tan\lambda$, where λ is the angle of friction, hence

$$\cot\theta = \frac{1 - \tan^2\lambda}{2\tan\lambda} = \cot 2\lambda$$

$$\therefore \theta = 2\lambda$$

Exercises (6)

(1) A uniform ladder of weight W rests at an angle 45° to the horizontal with its ends resting on a smooth floor and against a smooth vertical wall, the lower end being joined by a string to the junction of the floor and the wall. Find the tension of the string and the reactions at the wall and the floor, when a man, whose weight is that of the ladder, has ascended three quarters of the ladder's length.

(2) A ladder A B of weight 20 kg. rests at an angle 45° to the horizontal with its ends resting on a smooth floor and against a vertical wall, the lower end A being joined by a string to the junction of the floor end the wall. The string cannot stand a tension of more than 50 kg. wt. without breaking. A man of weight 60 kg. wt. ascended quarters of the length of the ladder and founded that the string is about to break. Find at what point of the ladder its weight acts.

(3) A uniform ladder AB of weight 20 kg.wt rests at an angle 45° to the horizontal, with its ends resting on a smooth floor and against a smooth vertical wall. The lower end A is attached by a string to the junction of the wall and the floor. Given that the string can withstand a tension of not more than 25 kg. wt. , prove that a man of the same weight as the ladder cannot ascend more than three-quarters of the length of the ladder without breaking the string.

(4) A ladder A B of weight 35 kg.wt and length 3m. rests in a vertical plane with the end B on a smooth floor and A against a smooth vertical wall. The lower end B is attached by a string to a point on the floor vertically below A. Given that B is 1.8 m. away from the wall and the weight of the ladder is acting at a point on the ladder 1.2 m. away from B, find the tension in the string. Also find the tension in the string when a man of weight 80 kg. stands at the mid-point of the ladder.

(5) A uniform bar A B, hinged at A to a vertical wall, weighs 4 Newtons, and is 120 cm. long. A weight of 3 Newtons is hung from a point on the bar 80cm. From A and the bar is kept horizontal by a string attached with one end to B and is fixed to the wall 160 cm. above A. Find the tension of the string and the reaction at the hinge.

(6) A uniform rod, hinged at A to a vertical wall, weighs 10 Newtons and is 200 cm. long. A weight equals to that of the rod is hung from the end B, and the rod is kept horizontal by a rope tied to a point on it 150 cm. from A and the other end is fixed to the wall at a point above A. If the rope is inclined at 30° to the horizontal, find the tension in the rope and the reaction at the hinge.

(7) A uniform rod A B, of weight 200 Newtons, is hinged at A to a vertical wall and carries a weight of 100 Newtons at B. The rod, inclined at 30° to the horizontal is supported by a string attached to the end B and fixed to wall at a point C vertically above A such that AC = AB = BC. Find the tension in the string and the reaction at the hinge.

(8) A uniform rod rests in a vertical plane with the upper end against a smooth vertical wall and the lower end on a horizontal rough ground. The coefficient of friction between the rod and the ground being equal to $\frac{1}{3}$. If the rod rests in limiting equilibrium, find the angle of inclination of the rod to the wall.

(9) A uniform ladder of weight 20 kg. rests in a vertical plane with one end on a rough horizontal floor and the other end against a smooth vertical wall. The ladder is inclined at 60° to the horizontal. Given that the coefficient of friction between the ladder and the floor equal $\frac{1}{2\sqrt{3}}$, prove that the maximum distance a man of weight 60 kg. can ascend up the ladder is equal to half length of the ladder.

(10) A uniform rod of weight 15 Newtons rests in a vertical plane with one end on a rough horizontal floor and the other end on a smooth vertical wall. If the ladder rests in limiting equilibrium when its inclination to

the horizontal is equal to $30°$. Find the coefficient of friction between the rod and the floor and find the reaction of the wall on its end.

(11) A uniform rod AB of weight W and length 260 cm. rests in a vertical plane with its ends resting on a rough horizontal floor and against a smooth vertical wall. The coefficient of friction between the rod and the floor equals $\frac{1}{2}$, and the lower end B of the rod is 100 cm. away from the wall. Find the horizontal force acting at B that will make the motion about to begin towards the wall.

(12) A uniform rod rests with one end against a rough vertical plane, the coefficient of friction between them is $\frac{1}{2}$, and the other end on a rough horizontal floor, the coefficient of friction between them is $\frac{3}{4}$. Find the rod's inclination to the floor when being in limiting equilibrium.

(13) A uniform rod of weight 40 Newtons rests in a vertical plane with one end against a rough vertical wall, the coefficient of friction between them is $\frac{1}{2}$, and the other end on a rough horizontal floor, the coefficient of friction between them is $\frac{1}{3}$. The rod is inclined at $45°$ to the floor. Find the least horizontal force that will make the lower end of the rod about to move towards the wall.

(14) A uniform ladder rests in a vertical plane with its ends on a horizontal floor and against a vertical wall. The ladder is inclined to the wall with an angle which tangent is $\frac{6}{11}$, and the coefficient of friction between the ladder and either the wall or the floor is $\frac{1}{3}$. If a man whose weight is three times as much as that of the ladder is ascending it, prove that he cannot exceed seven-tenths of the length of the ladder without disturbing equilibrium .

Chapter Five

Couples

Preface:

In this chapter we will deal with : definition of couple , equilibrium of a body under the act of two couples and the resultant of a set of couples.

Objectives:

By the end of teaching this chapter, the student should be able to:

(1) Recognize the concept of couple.

(2) Calculate the moment of a couple .

(3) Deduce that the moment of a couple is a constant vector .

(4) Recognize the concept of equilibrium of a body under the act of two coplanar couples.

(5) Find the resultant of set of couples.

(6) Solve life problems on couples.

Topics :

(1) Couple (concept - definition - moment) .

(2) Equilibrium of a rigid body under the act of two coplanar couples.

(3) The sum of two coplanar couples.

(4) The sum of a finite number of couples.

Couples

The concept of a couple is considered as one of the fundamental concepts of Mechanics. This chapter will be devoted to demonstrate the concept of a couple, and its important properties and the fundamental theorems concerned with it.

Definition :

A system of two forces of equal magnitudes and opposite direction and acting in different lines of action is called a COUPLE.

The last condition is very important in the definition of a couple, since if the two forces have the same line of action, the two forces will be in equilibrium, but the non-vanishing of the normal distance between the lines of action of the two forces, the two forces are not in equilibrium, as we notice that from our daily experience.

Figure (69) shows two examples of a couple.

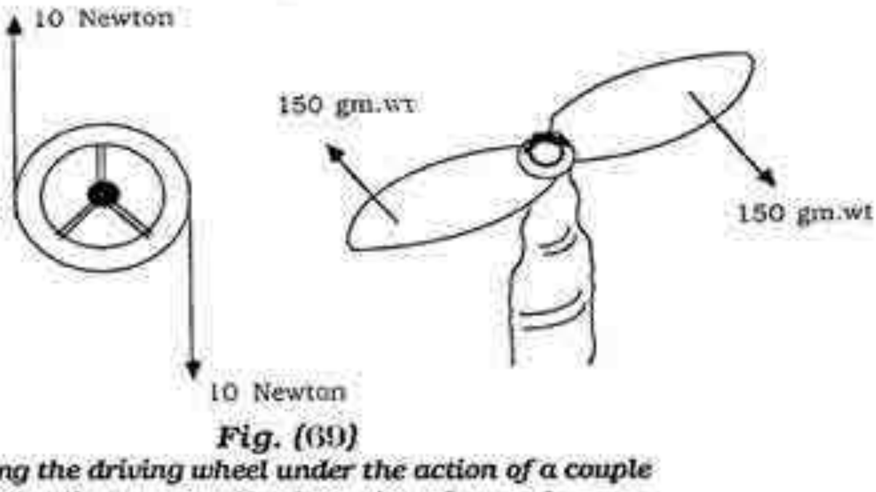


Fig. (69)
rotating the driving wheel under the action of a couple
rotating a tap under the action of a couple

Moment of a couple :

Let $\vec{F}$ and $-\vec{F}$ be the two forces forming the couple. $\|\vec{F}\| = F$

Draw the common normal to the lines of action of the two forces, let it meet them at the points A and B respectively, and p = AB be the length of the common normal to the lines of action Fig. (70).

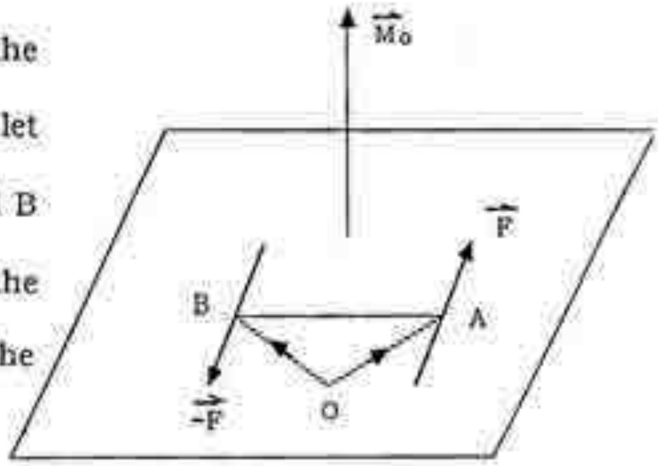


Fig. (70)

Calculate the sum of the moments of the two forces of the couple about an arbitrar point O.

$$\vec{M_0} = \vec{OA} \times \vec{F} + \vec{OB} \times (-\vec{F})$$

$$= (\vec{OA} - \vec{OB}) \times \vec{F}$$

$$= \vec{BA} \times \vec{F}$$ (refer to the triangle of vectors OAB in fig. (70)

Since the two points A, B, are independent of the position of the point O about which we are taking moments, the sum of moments of the two forces of the couple is independent of the position of O, and thus it is a constant vector called Moment of Couple and will be denoted by $\vec{M}$.

Therefore we have :

$\overrightarrow{M} = \overrightarrow{BA} \times \overrightarrow{F} = \overrightarrow{AB} \times (- \overrightarrow{F})$

which means that the moment of a couple is equal to the moment of one of the forces f the couple about a point on the line of action of the other force.

N.B.

The moment of a couple does not change if the point A is replaced by any other point on the line of action of the force $\overrightarrow{F}$, and the point B is replaced by any other point on the line of action of the force $(- \overrightarrow{F})$.

We formulate our results in the following fundamental theorem.

Theorem :

The moment of a couple is a constant vector, independent of the point about which we take the moments of the two forces and is equal to the moment of one of the forces of a couple about any point on the line of action of the second force.

Magnitude and direction of the moment of a couple :

Since the vector $\overrightarrow{BA}$ is perpendicular to the vector $\overrightarrow{F}$, the angle between these two vectors is 90°.

$\therefore \quad \| \overrightarrow{M} \| = \| \overrightarrow{BA} \| \; \| \vec{F} \| \sin 90°.$

$\therefore \quad \| \overrightarrow{M} \| = F\,p$

i.e. "The magnitude of the moment of a couple is equal to the product of the magnitude of one of the forces times the length of common normal to their lines of action".

N.B. :

The length of common normal to the lines of action of the two forces of a couple p is called "The arm of the couple". We notice that the moment of a couple is perpendicular to both $\overrightarrow{BA}$, $\vec{F}$. i.e. to the plane containing the lines of action of the two forces. The direction of this moment is determined according to the right hand rule.

Coplanar Couples :

If a number of couples act on a rigid body, and if the lines of action of the forces of these couples all lie in the same plane, it is said that these couples form a system of coplanar couples.

In what follows we are concerned only with coplanar couples. It is clear that the moments of a sstem of coplanar couples are all parallel and perpendicular to the plane of the forces fig. (71). This makes the study of the moments easier, since it will be possible to deal with the algebraic measures of these moments (relative to a unit vector parallel to it) instead of dealing with vector moments themselves.

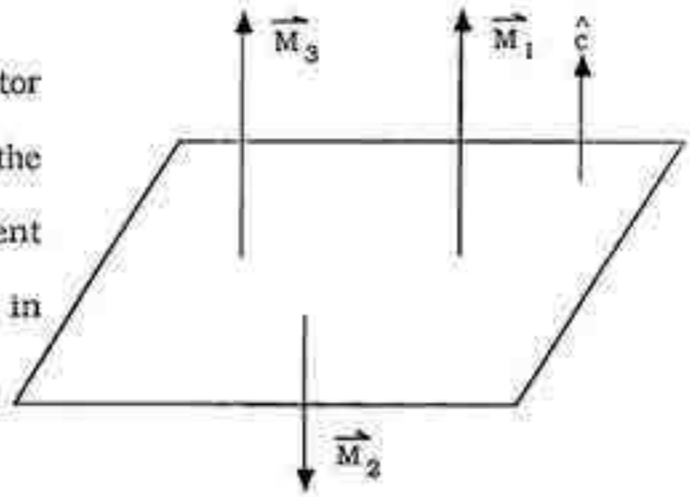


Fig. (71)

If we take $\hat{c}$ as a unit vector perpendicular to the plane of the forces, we can write the moment of any couple of the system in terms of $\hat{c}$ as follows :

$$\overrightarrow{M} = M\,\hat{c}$$

Where M is the algebraic measure of the moment $\overrightarrow{M}$ with respect to the unit vector $\hat{c}$. But since we are going to deal only with the algebraic measures of the monents of the coplanar couples, we will drop down the unit vector $\hat{c}$ and determine the sign of the algebraic measure to the moment of a couple according to the following rule :

Rule :

If when looking of the plane of the forces, we find that the couple is tending to a rotation in an anti-clockwise direction, the algebraic measure of its moment is considered positive Fig. (72-a), but if the couple is tending to a rotation in a clockwise direction, the algebraic measure of its moment is considered negative fig. (72-b).

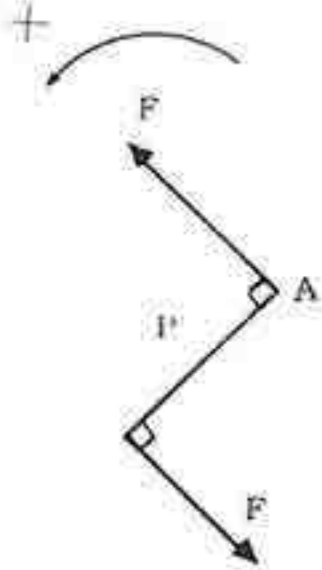


Fig. (72-a)

couple tending to a rotation

in an anti-clockwise direction

$\overrightarrow{M}$ is in direction of $\hat{c}$, $M > 0$

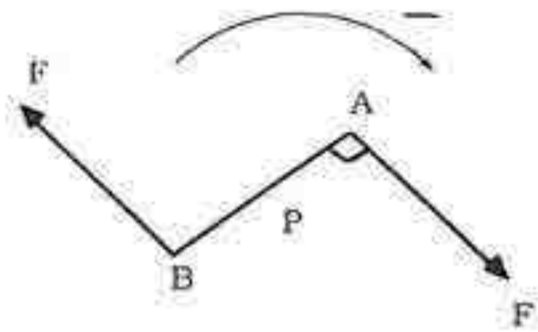


Fig. (72-b)

couple tending to a rotation

in a clockwise direction

$\overrightarrow{M}$ in an opposite direction to

$\hat{c}$, $M < 0$

THE EQUILIBRIUM OF A RIGID BODY UNDER THE EFFECT OF TWO COOPLANER COUPLES

Experiment : Equilibrium of two couples.

Aim of experiment :

To show that if a rigid body is in equilibrium under the action of two coplanar couples, then the amounts of these couples are qual in magnitude, and opposite in direction.

Apparatus :

The apparatus consists of a holder, two reels, a ruler with holes, a sensitive spring balance, three weight carriers, pivot, strings, a small metal rider.

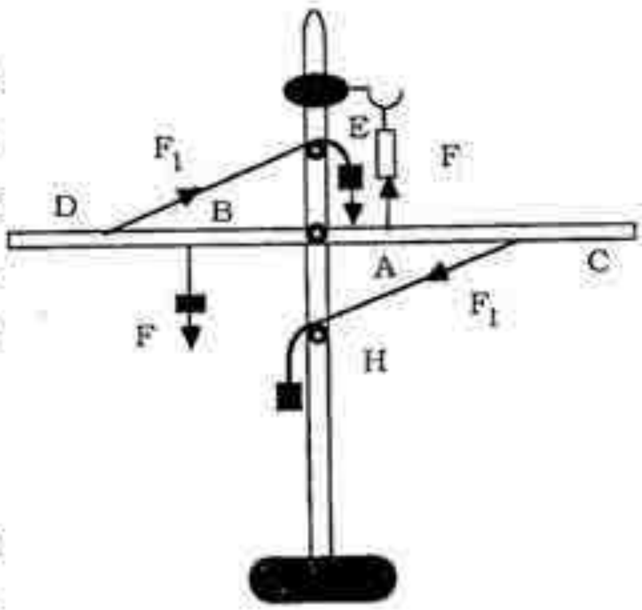


Fig. (73)

Working steps :

1) Fix two equal reels in the holder so that their centres are at equal distances from the pivot.

2) Put the ruler on the pivot at its midpoint, and if the ruler is not in equilibrium in a horizontal position use the metal rider and move it on the ruler until the ruler is in equilibrium in a horizontal position. Be sure that the ruler is horizontal by using a level balance.

3) Fix a string at A to a balance fixed in the holder so that the string is vertical.

4) Fix a string carrying a weight at B, where OA = OB, O being the pivot.

5) Take two points such as C, D at equal distances from O i.e. OC = OD.

6) Fix in each of these points a string passing over a reel, and carrying weights as shown in figure.

7) Put equal weights on the carriers at E, H, until the ruler is in equilibrium in a horizontal position.

8) Compare the weight suspended at B, and the tension in the balance, you will find that they are equal, let the magnitude of each be F.

9) Find each of the two equal forces acting at C, D, let each be equal to F_1.

10) Find the distance between the two equal forces F_1 and let it be P.

11) Fix the end of a string at the pivot O and move the string so as to measure the shortest distance between O and any of the two inclined strings CH & DE as in fig. (73), which is half the distance between them. Let the distance between the two inclined strings be P_1. Compare the two products $F \times p$ and $F_1 \times P_1$, you will find that they are equal. Repeat the experiment

Several times by changing the values of F and F1. Find the percentage error in the results of the experiment.

From the previous experiment, we can give the following definition for the equilibrium of a rigid body under the effect of two cooplaner couples.

A rigid body is said to be in equilibrium under the effect of two cooplaner couples if the sum of their moments is the zero vector.

If $\overrightarrow{M_1}$, $\overrightarrow{M_2}$ are the moments of the two couples, then the condition of the equilibrium of a rigid body under the effect of two cooplaner couples is written in the form.

$$\overrightarrow{M_1} + \overrightarrow{M_2} = \overrightarrow{O}$$

Or $\overrightarrow{M_1} = - \overrightarrow{M_2}$

Since $\overrightarrow{M_1} = M_1 \; \hat{C}$, $\overrightarrow{M_2} = M_2 \; \hat{C}$

Where M_1, M_2 are the algebraic measures of the moment vectors $\overrightarrow{M_1}$, $\overrightarrow{M_2}$ respectively relative to the unit vector , therefore

$$\overrightarrow{M_1} + \overrightarrow{M_2} = M_1 \; \hat{C} + M_2 \; \hat{C}$$

$$= (M_1 + M_2) \; \hat{C}$$

Thus the sum $(\overrightarrow{M_1} + \overrightarrow{M_2})$ vanishes if the sum of the algebraic measures $(M_1 + M_2)$ vanishes and vice versa. We thus obtain the following result :

Result :

A rigid body is said to be in equilibrium under the effect of two cooplaner couples if the algebraic sum of their moments vanishes.

i.e. $\boxed{M_1 + M_2 = 0}$

Example (1) :

$\overline{AB}$ is a rod of negligible weight and of length 100 cm. C, D are two points on the rod 40, 80 cm, distant from the end A respectively. Forces of magnitudes 300, F, 300, F newtons act at the points A, C, D, B respectively in directions perpendicular to the rod, such that the two forces at A, B are in the same direction, and the other two forces in the opposite direction. Find the value of F if the rod is in equilibrium.

Solution :

The rod is in equilibrium under the action of two couples: a couple formed of the two forces 300, 300 newtons at A, D. Let M_1 be the algebraic measure of its moment and another couple formed of the two forces F , F newtons at C, B. Let M_2 be the algebraic measure of its moment. Referring to fig (74).

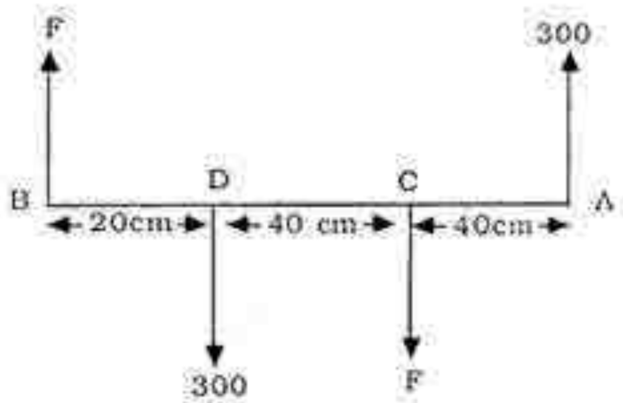


Fig. (74)

$\therefore\ M_1 = 300 \times 80 = 24000$ newton. cm.

& $M_2 = -F \times 60 = -60\ F$ newton. cm.

Since the rod is in equilibrium, the two couples must be in equilibrium.

$\therefore\ M_1 + M_2 = 0$ $\qquad\qquad$ $\therefore\ 24000 - 60\ F = 0$

$$\therefore\ F = \frac{24000}{60} = 400 \text{ newtons}$$

Example (2) :

$\overline{AB}$ is a rod of negligible weight. It is suspended in a horizontal position by a pin at its midpoint. Two forces each of magnitude 7.5 newtons act at its ends, that at A is vertically downwards and that at B is vertically upwards. It is also pulled by a string in a direction making 60° with the rod from a point C on it. Find the magnitude and direction and the point of action of the force which if it acts with the other forces on the rod will deep it in equilibrium in a horizontal position, given, that the tension in the string is of magnitude 10 newtons and the length of the rod is 30 cm.

Solution :

The two forces 7.5, 7.5 newtons at A, and B form a couple, the algebraic measure of its moment is $M_1 = -7.5 \times 30 = -225$ newton. cm.

Since the rod is to be in equilibrium, it must be affected by another couple has a moment of the same magnitude and of opposite direction.

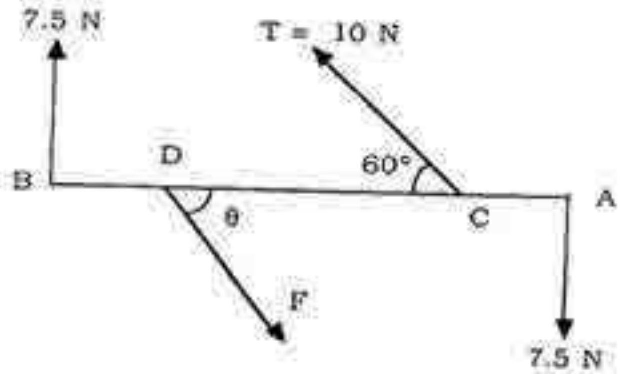


Fig. (75)

then the tension $\overrightarrow{T}$ and the force $\overrightarrow{F}$ will form a couple of opposite moment to the moment of the first couple.

$\therefore \quad F = T = 10$ newtons, $\theta = 60^\circ$

The algebraic measure of the moment of this couple

$M_1 = 10 \times CD \sin 60^\circ = 5\sqrt{3}$ CD.

$\therefore \; M_2 + M_2 = 0 \qquad \therefore \; 5\sqrt{3}$ CD $= 225$

$$\therefore \; CD = \frac{225}{5\sqrt{3}} = 15\sqrt{3} \text{ cm.}$$

i.e. D is at a distance of $15\sqrt{3}$ cm from C.

Example (3) :

AB is a uniform rod of length 60 cm, and weight 10 kg. wt acting at its midpoint, moves in a vertical plane about a fixed hinge at its end A. A couple in a vertical plane and of moment 150 kg. wt. cm acts on the rod. Prove that the reaction of the hinge at A is equal to the weight of the rod, and find the inclination of the rod to the horizontal in equilibrium position.

Solution :

Since the force acting on the rod, are its weight, the reaction of the hinge at A, and the couple, so in order that the rod is to be in equilibrium, it must be affected by another couple has a moment of the same magnitude and of opposite direction, so the weight and the reaction form a couple.

Thus the reaction at A will act vertically upwards and will be equal in magnitude to the weight of the rod, i.e. the reaction at A will be of magnitude 10 kg. wt.

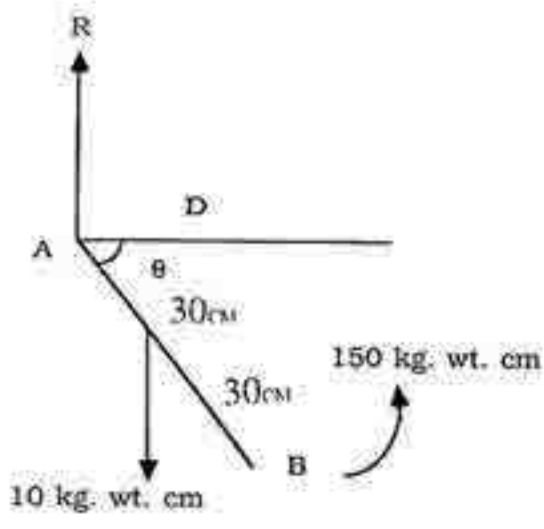


Fig. (76)

The algebraic measure of the moment of the couple formed of the two forces, the weight and the reaction $M_1 = -10 \times 30 \cos\theta$

$M_1 = -300 \cos\theta$ kg. wt. cm

The algebraic measure of the given couple $M_2 = 150$ kg. wt.cm.

In equilibrium $M_1 + M_2 = O$

$\therefore \quad 150 - 300 \cos \theta = 0$

$\therefore \quad \cos \theta = 1/2$. $\qquad \therefore \quad \theta = \pm 60°$

i.e. there are two positions of equilibrium, in which the rod is inclined at an angle 60° to the horizontal either upwards or downwards.

Example (4) :

ABCD is a fine lamina of weight 3 newtons in the form of a square the length of whose side is 50 cm. It has a small hole near the vertex A, and is susperrded from this hole by a thin pin, so that its plane is vertical. Find the pressure of the pin. If a couple of moment 7.5 newton.cm acts on the lamina in its plane, prove that the pressure on the pin does not change, then find the inclination of the diagonal $\overline{AC}$ to the vertical in the equilibrium position, given that the weight of the lamina acts at the point of intersection of the diagonals.

Solution :

(1^{st}) In the equilibrium position, the lamina is under the action of two forces, its weight acting at the point of intersection of the diagonals O.

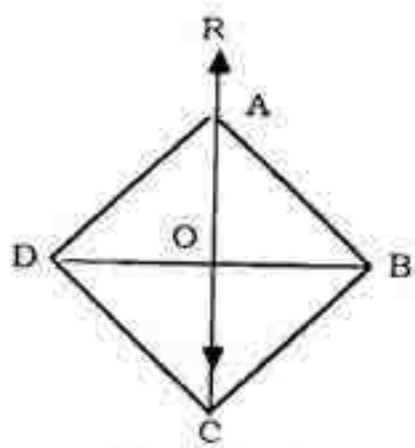


Fig. (77 - a)

and the reaction at A, and thus the reaction at A is a vertical force action upwards and of magnitude 3 newtons since the body is in equilibrium under the action of two forces only, and thus they must be equal in magnitude, have the same line of actin and in opposite directions.

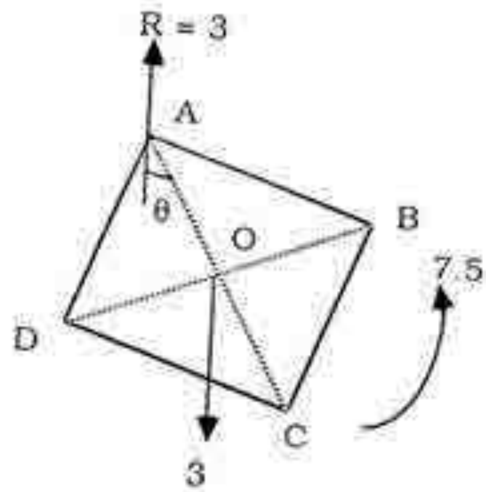


Fig. (78 - b)

(2^{nd}) In equilibrium position the lamina is under the actin of two forces, the weight of the lamina, the reaction at A, together with the couple as in figure (78.b)

So, in order that the body is to be in equilibribum, it must be effected by another couple has a moment of the same mangitude and in the opposite direction.

Therefore the reaction at A is a vertical force of magnitude 3 newtons acting vertically upwards.

Also $7.5 - 3\,(AO \sin\theta) = 0$

$\therefore\ AO = 25\sqrt{2}$ cm.

$\therefore\ 7.5 = 3 \times 25\sqrt{2}\ \sin\theta.$

$$\therefore \sin\theta = \frac{\sqrt{2}}{20} = \frac{1.414}{20} = 0.0707 \quad \therefore \theta = 4^\circ\ 3'\ 15''$$

Equivalence of two couples :

Definition :

Two couples in the same plane are equivalent if the algebraic measures of their moments are equal.

Proof :

$\therefore \overrightarrow{M_1} = M_1\ \hat{c}$, $\overrightarrow{M_2} = M_2\ \hat{c}$

The condition for equivalence is $M_1\ \hat{c} = M_2\ \hat{c}$

$$\therefore M_1 = M_2$$

Example (5) :

$\overline{AB}$ is a rod of negligible weight and length 1.5 m. two equal forces each of magnitude 200 newtons act at its point of trisection in two opposite directions perpendicular to the rod. If these two forces are replaced by two other forces each of magnitude 120 newtons acting at the ends of the rod such that they form a couple equivalent to the first couple, what is the inclination of the lines of action of these two forces to the rod.

Solution :

Let the two forces 200, 200 N act in the two directions shown in fig. (80).

The algebraic measure of the couple formed :

$M_1 = -200 \times 0.5 = -100$ newton. m.

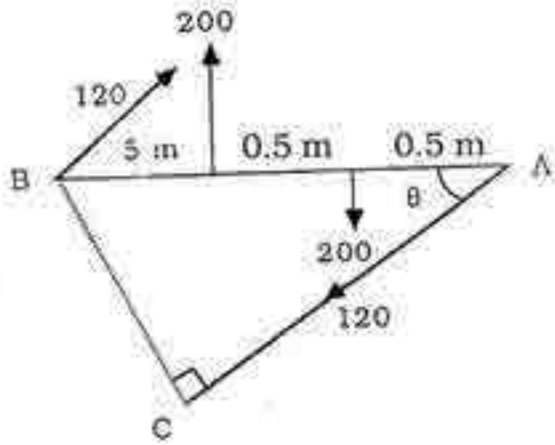


Fig. (80)

The negative sign shows that this couple tends to a rotation in a clockwise direction.

Since the algebraic measure of the new couple is equal to M_1 the two forces 120, 120 N. act in the shown directions in figure.

To calculate M_2 , draw the perpendicular from B on the line of action of the force acting at A to meet it at C say, let θ be the inclination of each of the two forces to the rod.

$\therefore M_2 = -120 \times (AB \times \sin\theta)$

$= -120 \times (1.5 \times \sin\theta) = -180 \sin\theta$

$\therefore M_1 = M_2$ $\qquad \therefore -180 \sin\theta = -100$

$$\sin\theta = \frac{100}{180} = \frac{5}{9}$$

$\therefore \theta = 34'$

Example (6) :

ABCD is a square of side length 1 m. Two forces each of magnitude 4 kg. wt act along $\overrightarrow{AB}$, $\overrightarrow{CD}$, also two other forces each of magnitude F kg. wt act at B and D so that the first force makes an angle of measure 15° with $\overrightarrow{DA}$, the second force makes an angle of measure 50° with $\overrightarrow{BC}$. Find the value of F if the couple formed by the first two forces is equivalent to the couple formed by the two last forces.

Solution :

It is clear that the moments of the two couples are in the same direction. because each is tending to a rotation in a clockwise direction relative to a viewer looking at the figure.

The magnitude of the moment of the first couple

$M = 4 \times 1 = 4$ kg. wt. m.

To calculate the magnitude of the moment of the second couple, draw the perpendicular from D on the line of action of the force F acting at B to meet it at E say.

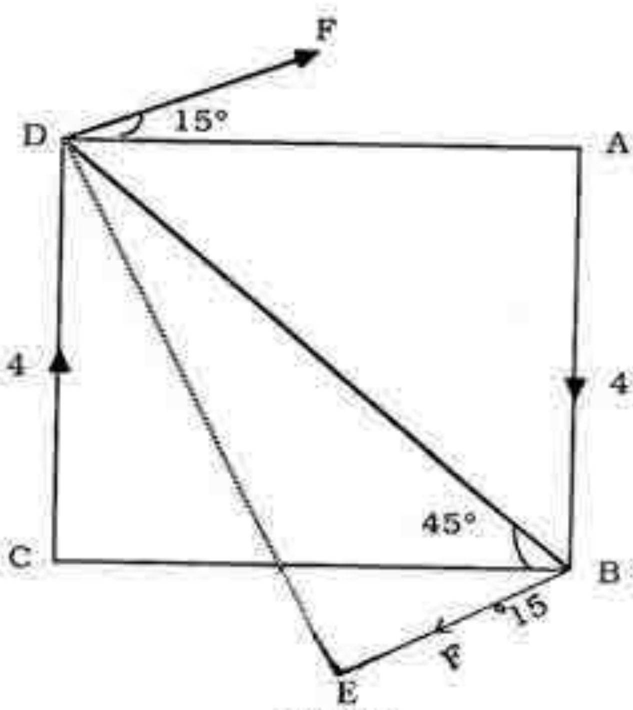


Fig. (81)

$DE = DB \sin (15° + 45°) = DB \sin 60°$

But $DB = \sqrt{2}$ m. $\quad\therefore\ DE = \sqrt{2} \times \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{\sqrt{2}}$ m.

$\therefore$ The magnitude of the moment of the second couple

$M_2 = F \times DE = \frac{\sqrt{3}}{\sqrt{2}} F$ Kg. wt. m.

The condition for the equivalence of the two couples :

$\frac{\sqrt{3}}{\sqrt{2}} F = 4 \quad\quad \therefore F = \frac{4\sqrt{2}}{\sqrt{3}}$ kg. wt.

$= \frac{4}{3} \sqrt{6}$ kg. wt.

Exercises (5 - 1)

1) AB is a rod of negligible weight and of length 60 cm. Four parallel forces and normal to the rod act on it at the points and in the directions indicated in the figures (82 - a, b, c). The magnitudes of the force are all related to the same units of magnitude of a force.

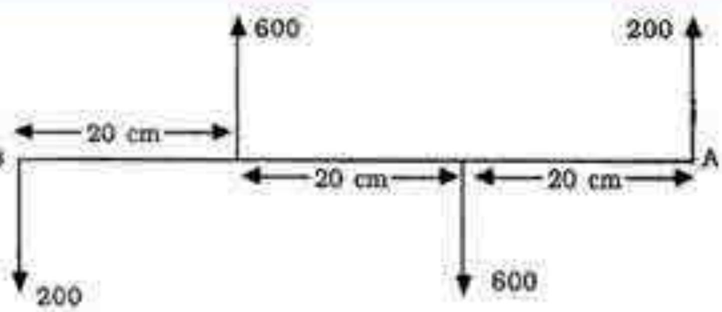


Fig. (82 - a)

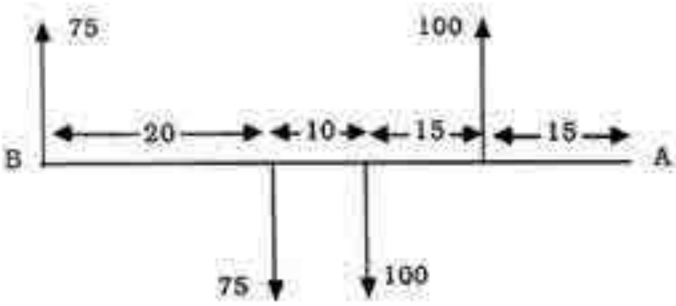


Fig. (82 - b)

Prove that the body is in equilibrium in the two cases (a) and (b) only.

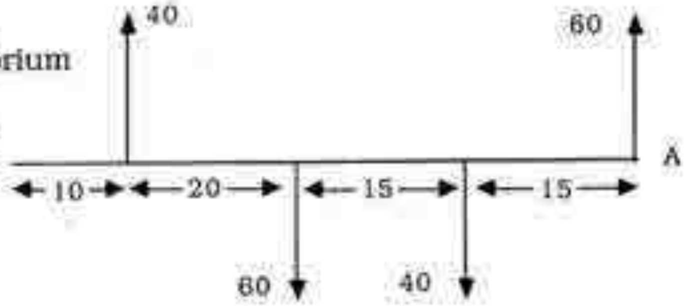


Fig. (82 - c)

2) Two coplanar couples act on a rod AB of negligible weight and of length 90 cm. The first couple consists of the two forces F, F kg.wt, and the second consists of the two forces, 2, 2 kg. wt acting at the points, and the directions shown in fig. (83).

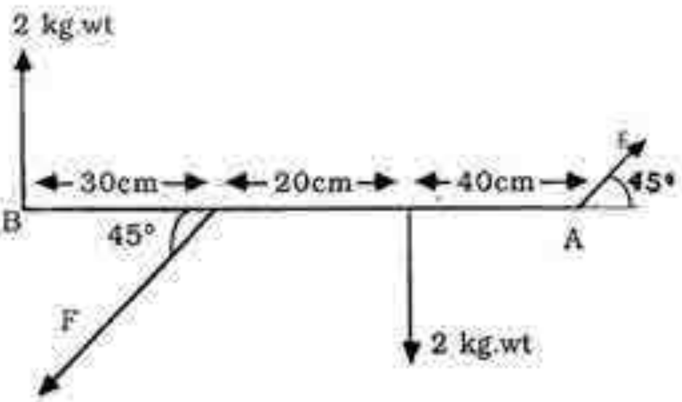


Fig. (83)

Find the value of F which makes the body to be in equilibrium under the two couples.

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3) ABCD is a rectangle in which AB = 40 cm. BC = 30 cm. Two forces each of magnitude 200 newtons act along $\overrightarrow{AB}$ and $\overrightarrow{CD}$. Another two equal forces F and F act in A and C and parallel to $\overleftrightarrow{BD}$. Determine F to make the two couples be equivalent.

4) A rod of length 40 cm, and of weight 2.4 kg.wt acting at its midpoint, can rotate easily in a vertical plane about a fixed hinge at its end. A couple of magnitude 24 kg. wt. cm and whose direction is perpendicular to the vertical plane in which the rod can rotate, acts on the rod. Find the magnitude and direction of the reaction of the hinge, and the inclination of the rod to the verical in the position of equilibrium.

5) AB, a rod of length 60 cm and of weight 18 N. acting at its mid-point can rotate easily in a vertical plane about a horizontal pin passing through a hole in the rod at the point C 15 cm distant from A. If the rod rests with its end B on a smooth horizontal table, and the end A is pulled with a rope, horizontally until the reaction of the table becomes equal to the weight of the rod, find the tension in the rope, and the reaction of the pin given that the rod is in equilibrium in a position in which it is inclined at

an angle of measure $60°$ to the horizontal. Find also the magnitude and direction of the reaction of the pin.

6) ABCD is a fine lamina in the form of a square, of side length 50 cm and weight 300 gm. wt acting at the intersection of its diagonals. A small hole is made in the lamina near A, and the lamina is suspended by a thin horizontal pin through the hole, so that it is in equilibrium in a vertical plane. Find the pressure on the pin. If a couple, the magnitude of its moment is 7500 gm. wt. cm act in a direction perpendicular to the plane of the lamina, prove that the pressure on the pin does not change, and find the inclination of the diagonal $\overline{AC}$ to the vertical in the position of equilibrium.

7) ABCD is a fine lamina in the form of a square, of side length 20 cm, and weight 150 N, acting at the intersection of its diagonals. The lamina is suspended by a thin horizontal pin passing through a small hole near the vertex D, it is in equilibrium in a vertical plane, find the pressure on the pin. If a couple acts on te lamina, so that its direction is perpendicular to its plane, find the magnitude of the moment of the couple if the lamina is in equilibrium with $\overline{AD}$ horizontal.

8) ABCD is a fine lamina in the form of a rectangle in which AB = 18 cm, BC = 24 cm and of weight 20 N. acting at the intersection of its diagonals. The lamina is suspended by a thin horizontal pin passing through a small hole near the vertex D, so that its plane is vertical. If a couple, the magnitude of its moment is 150 N. cm and its direction is perpendicular to the plane of the lamina, acts on the lamina, find the inclination of $\overline{DB}$ to the vertical in the position of equilibrium.

9) ABC is a fine lamina in the form of a right-angled triangle at B, AB = 12 cm, BC = 15 cm, and its weight is 6 newtons acting at the intersection of its medians. If a couple whose direction is perpendicular to the plane of the lamina acts on it so that it is in equilibrium in a position in which $\overline{AB}$ is vertical, find the magnitude of the moment of the couple.

10) ABC is a lamina in the form of an equilateral triangle, its wight is 50 gm. wt acting at the point of intersection of its medians. The lamina is suspended by a thin horizontal pin passing through a small hole near the vertex A, so that its plane is vertical. A couple whose direction is perpendicular to the plane of the lamina, and the magnitude of its moment is 250 gm. wt. cm acts on it. If the lamina is in equilibrium, find the inclination of $\overline{AB}$ to the horizontal, given that the height of the triangle is 15 cm.

The sum of two coplanar couples :

Consider two coplanar couples, the first is composed of the two forces $\overrightarrow{F_1}$, $-\overrightarrow{F_1}$, and the second is composed of the two forces $\overrightarrow{F_2}$, $-\overrightarrow{F_2}$ Their moments are $\overrightarrow{M_1}$, $\overrightarrow{M_2}$ respectively. The four forces $\overrightarrow{F_1}$, $-\overrightarrow{F_1}$, $\overrightarrow{F_2}$, $-\overrightarrow{F_2}$ all lie in te same plane.

Let $\overrightarrow{R}$ be the resultant of the two forces $\overrightarrow{F_1}$, $\overrightarrow{F_2}$.

$\therefore \quad \overrightarrow{R} = \overrightarrow{F_1} + \overrightarrow{F_2}$.

Le $\overrightarrow{R'}$ be the resultant of the two forces $-\overrightarrow{F_1}$, $-\overrightarrow{F_2}$.

$$\therefore \quad \overrightarrow{R'} = (-\overrightarrow{F_1}) + (-\overrightarrow{F_2}) = -\overrightarrow{F_1}, -\overrightarrow{F_2}.$$

$$= -(\overrightarrow{F_1} + \overrightarrow{F_2})$$

$$= -\overrightarrow{R}.$$

Therefore it is possible to reduce the four forces into two parallel forces $\overrightarrow{R}$, $-\overrightarrow{R}$ which form a couple in the same plane as the two couples.

If $\overrightarrow{M}$ is the moment of this coupe, then

$\overrightarrow{M}$ = (Moment of the force $\overrightarrow{R}$ about O) + (Moment of the force $-\overrightarrow{R}$ about O).

= (Moment of the force $\overrightarrow{F_1}$ about O + Moment of the force $\overrightarrow{F_2}$ about O)

+ (Moment of the force $-\overrightarrow{F_1}$ about O + Moment of the force $-\overrightarrow{F_2}$ about O).

= (Moment of $\overrightarrow{F_1}$ about O + Moment of $-\overrightarrow{F_1}$ about O)

+ (Moment of $\overrightarrow{F_2}$ about O + Moment of $-\overrightarrow{F_2}$ about O)

= $\overrightarrow{M_1} + \overrightarrow{M_2}$

According to this we give the following definition to the sum of two coplanar couples :

Definition :

The sum of two coplanar couples is defined as the couple whose moment is equal to the sum of the moments of the two couples.

$$\overrightarrow{M} = \overrightarrow{M_1} + \overrightarrow{M_2}$$

The sum of two coplanar couples is called the "Resultant couple" and it is said that we have reduced the two couples to a single resultant couple.

Result :

The algebraic measure of the moment of the sum of two coplanar couples is equal to the sum of the algebraic measures to their moments.

Proof :

Let $\overrightarrow{M_1} = M_1 \hat{c}$, $\overrightarrow{M_2} = M_2 \hat{c}$ be the moments of the two coplanar couples required to find their sum.

The moment of the sum is $\overrightarrow{M} = \overrightarrow{M_1} + \overrightarrow{M_2}$

$$= M_1 \hat{c} + M_2 \hat{c}$$

$$= (M_1 + M_2) \hat{c}$$

This means that the vector $\overrightarrow{M}$ is parallel to both the vectors $\overrightarrow{M_1}, \overrightarrow{M_2}$. Therefore if M is the algebraic measure of the vector $\overrightarrow{M}$. i.e. if $\overrightarrow{M} = M \hat{c}$

Comparing the last two relations, we have

$$\boxed{M = M_1 + M_2}$$

Example (1) :

ABCD is a square, of side length 30 cm. Two forces each of magnitude 200 gm. wt act along $\overrightarrow{AB}$, $\overrightarrow{CD}$ and two forces each of magnitude 500 gm. wt act along $\overrightarrow{AD}$, $\overrightarrow{CB}$. Find the algebraic maesure of the moment of the resultant couple.

Solution :

Let M_1 , M_2 be the algebraic measures of the moments of the couple formed of the two forces 200, 200 gm. wt, and the couple formed of the two forces 500, 500 gm. wt respectively, M the algebraic measure of the moment of the resultant couple

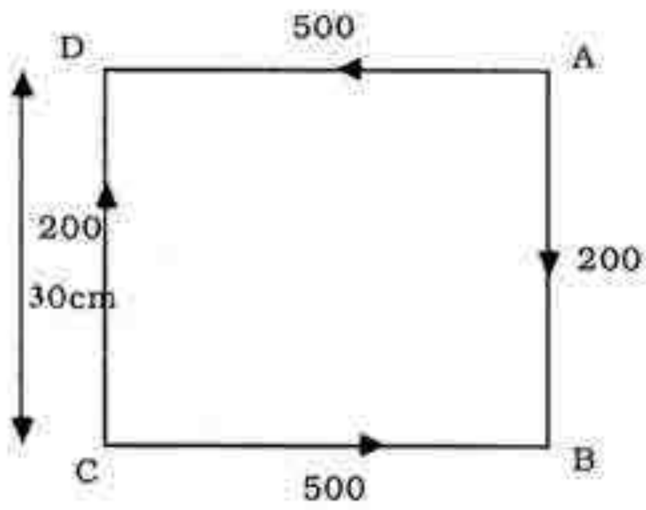


Fig. (84)

Referring to fig. (84) we have

$M_1 = -200 \times 30 = -6000$ gm. wt. cm.

$M_2 = -500 \times 30 = -15000$ gm. wt. cm.

$\therefore \quad M = M_1 + M_2 = -6000 + 15000$

$= 9000$ gm. wt. cm.

Example (2) :

AB is a rod of negligible weight and length 80 cm. The rod is under the action of :

Two forces each of magnitude 2 kg. wt., in opposite directions, acting at the end A and at O the midpoint of the rod, such that the force acting at A makes 30° with $\overrightarrow{AB}$.

A couple whose direction is perpendicular to the plane of the two forces, and the magnitude of its moment is 80 kg. wt.

Find the resultant couple if the moment of the given couple is in the same direction as that of the couple formed of the two forces and in the case in which it has the opposite direction.

Solution :

Let M_1 be the algebraic measure of the moment of the couple formed of the two given forces, M_2 the algebraic measure of the moment of the given couple, (the magnitude of his moment is 80 kg. wt. cm.). M the algebraic measure of the moment of the resultant couple.

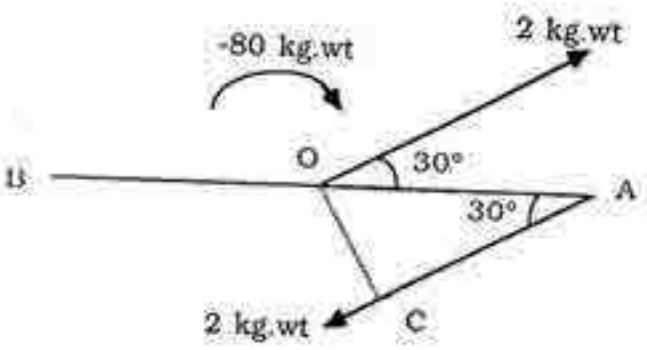


Fig. (85 - a)

To find M_1 we draw a perpendicular from O to the line of action of the force acting at A to meet it at C say. Noticing the moment's sign fig. (85 - a)

$M_1 = -2 \times OC$

$= -2 \times AO \sin 30^\circ$

$= -2 \times 40 \times 1/2 \quad = -40$ kg. wt. cm.

1st case :

If the moment of the given couple is in the same direction as the moment of the couple formed by the two given forces, as in fig. (89-a), the sign of M_2 will be as the sign of M_1 .

$\therefore \quad M_2 = -80$ kg. wt. cm.

$\therefore \quad M = M_1 + M_2$

$= -40 - 80 = -120$ kg. wt. cm.

2nd case :

If the moment of the given couple is opposite in direction to the moment of the couple formed by the two given forces, as in fig. (85-b), the sign of M_2 will be opposite to the sign of M_1.

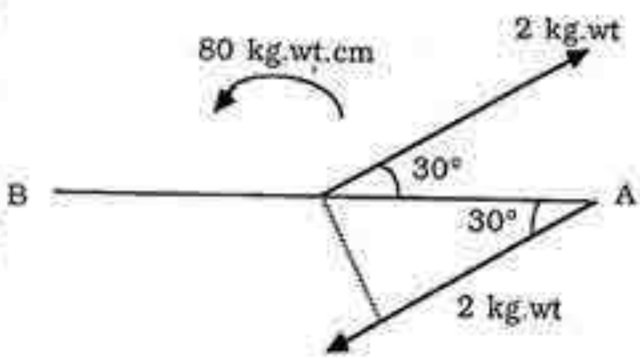


Fig. (85- b)

$\therefore M_2 = -80$ kg. wt. cm.

$\therefore M = M_1 + M_2$

$= -40 + 80 = 40$ kg. wt. cm.

Sum of any finite number of coplanar couples :

We can apply the process of obtaining the sum of two coplanar couples in obtaining the sum of any finite number of coplanar couples.

Definition :

The sum of any finite number of coplanar couples is defined as the couple whose moment is equal to the sum of the moments of these couples.

$$\boxed{\overrightarrow{M} = \overrightarrow{M_1} + \overrightarrow{M_2} + \dots\dots\dots + \overrightarrow{M_n}.}$$

Result :

The algebraic measure of the moment of the sum of several coplanar couples is equal to the sum of the algebraic measure of their moments.

$$M = M_1 + M_2 + \dots\dots\dots + M_n$$

Example (3) :

ABCD is a square, of side length 60 cm. Forces of magnitudes 40, 50, 40, 50 newtons act in the directions $\overrightarrow{AB}$, $\overrightarrow{CB}$, $\overrightarrow{CD}$, $\overrightarrow{AD}$, respectively. Two forces each of magnitude $30\sqrt{2}$ newtons act at A, C in the shown directions in fig. (86)

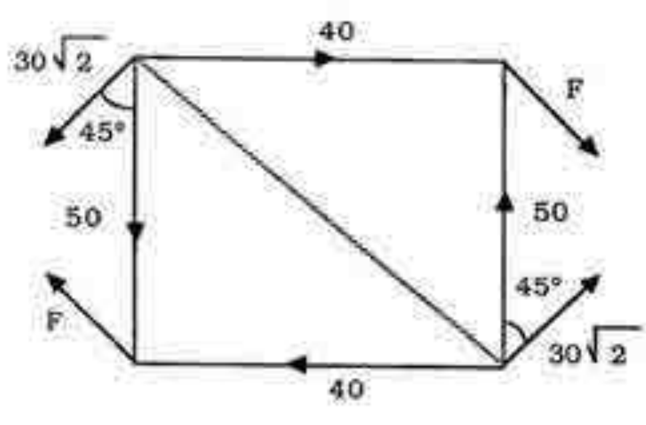


Fig. (86)

Find :

1) The couple equivalent to the system.

2) The magnitude and direction of the two forces acting at B. D and which are parallel to $\overleftrightarrow{AC}$ such that the system will be in equilibrium

Solution :

The two forces 40, 40 newtons form a couple, let M_1 be the algebraic measure of its moment.

$\therefore \quad M_1 = -40 \times 60 = -2400$ newton. cm.

The two forces 50, 50 newtons form a couple, let M_2 be the algebraic measure of its moment.

$\therefore \quad M_2 = 50 \times 60 = 3000$ newton. cm.

Also the two forces $30\sqrt{2}$, $30\sqrt{2}$ newtons form a third couple, let M_3 be the algebraic measure of its moment.

$\therefore \quad M_3 = 30\sqrt{2} \times Ac = 30\sqrt{2} \times 60\sqrt{2} = 3600$ newton. cm.

The system is equivalent to a couple, which is the sum of the three couples.

Let M be the algebraic measure of the sum of the moments of the three couples.

$M = M_1 + M_2 + M_3$

$= -2400 + 3000 + 3600 = 4200$ newton. cm.

Since M = O, the resultant couple is tending to a rotation in an anti-clockwise direction, and so if it is required that this couple be in equilibrium with the couple formed by the two forces $\vec{F}$, $-\vec{F}$ at B, D, the last couple must tend to a rotation in a clockwise direction. i.e. these two forces must be directed as shown in fig. (90), and the algebraic measure of its moment is negative.

Condition of equilibrium : $(-F \times BD) + 4200 = 0$

$\therefore F \times 60\sqrt{2} = 4200$

$\therefore F = \dfrac{4200}{60\sqrt{2}} = \dfrac{70}{\sqrt{2}} = 35\sqrt{2}$ newtons

Rule :

If three forces act on a rigid body and are completely represented by the sides of a triangle, taken the same way round, then this system is equivalent to a couple, the magnitude of its moment is equal to twice the area of the triangle divided by the drawing scale

Proof :

The directed straight Protions. $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{CA}$ represents the three forces completely i.e in magnitude, direction and line of action fig. (87). Let the magnitude of a force be represented by a drawing scale : 1 unit of lenght to each unit of magnitude of a force.

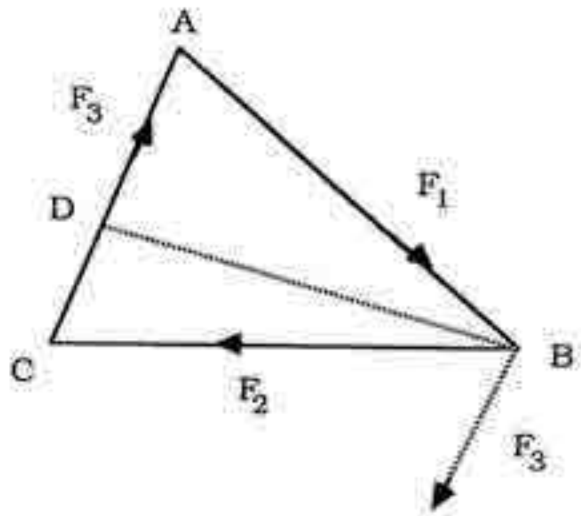


Fig. (87)

$\therefore \quad \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \overrightarrow{0}$

$\therefore \quad \overrightarrow{F_1} + \overrightarrow{F_2} + \overrightarrow{F_3} = \overrightarrow{0}$ i.e. $\overrightarrow{F_1} + \overrightarrow{F_2} = -\overrightarrow{F_3}$

On the other hand, the resultant of the two forces $\overrightarrow{F_1}$, $\overrightarrow{F_2}$ which meet at B passes through this point.

$\therefore$ The resultant of the two forces $\overrightarrow{F_1}$, $\overrightarrow{F_2}$ is a force $(-\overrightarrow{F_3})$ acting at B.

Thus the original system of the forces $\overrightarrow{F_1}$, $\overrightarrow{F_2}$, $\overrightarrow{F_3}$ is equivalent to the two forces, $\overrightarrow{F_3}$ acting at C and $(-\overrightarrow{F_3})$ acting at B. i.e. it is equivalent to a couple.

To find the magnitude of the moment of this couple, draw the perpendicular from B to $\overleftrightarrow{AC}$ to meet it at D say.

$\therefore$ Magnitude of moment of the couple = $\| \overrightarrow{F_3} \| \times BD$ but $\| \overrightarrow{F_3} \| = AC \times m$ where $\frac{1}{m}$ is the drawing scale

$\therefore$ Magnitude of moment of the couple = $AC \times m \times BD$ =

$$= (AC \times BD) \times m$$

$$= \frac{\text{twice the area of the triangle ABC}}{\frac{1}{m}}$$

$$= \frac{\text{twice the area of the triangle ABC}}{\text{drawing scale to the magnitude of the force}}$$

General theorem

If many coplanar forces act on a rigid body and are completely represented by the sides of a polygon, taken the same way round, then this system is equivalent to a couple, the magnitude of its moment is equal to twice the area of the polygon divided by the drawing scale.

Example (4) :

Three forces are completely represented by the sides of right – angled triangle ABC at B, taken the same way round, drawing scale being 1 cm to every 10 gm. wt.

Find the magnitude of the resulting couple given that AB = 40cm, BC = 30 cm.

Solution :

The magnitude of the resulting couple is twice the area of the triangle divided by the drawing scale.

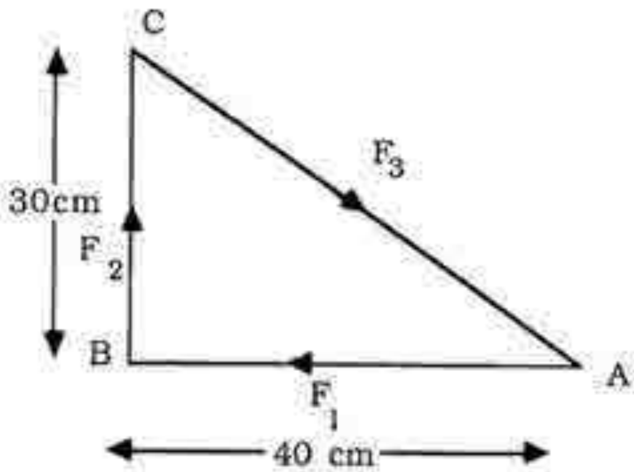


Fig. (88)

Twice The Area of triangle =

$= 30 \times 40$

$= 1200\ cm^2$

Drawing scale $= \dfrac{1\ cm}{10\ gm.\ wt.}$

$\therefore$ Magnitude of moment of couple

$= \dfrac{1200\ cm^2}{0.1\ cm\ /\ gm.\ wt.}$

$= 1200 \times 10 = 12000$ gm. wt. cm.

Example (5) :

ABCD is a trapezium in which m $(\angle A)$ = m $(\angle B)$ = 90°, AD = 10 cm, AB = 12 cm, BC = 15 cm. Forces of magnitudes 20, 24, 30, 26 newtons acted along $\overrightarrow{DA}$, $\overrightarrow{AB}$, $\overrightarrow{BC}$ and $\overrightarrow{CD}$ recpectively. Prove that the system of these forces is equivalent to a couple.then find the norm of its moment.

Solution :

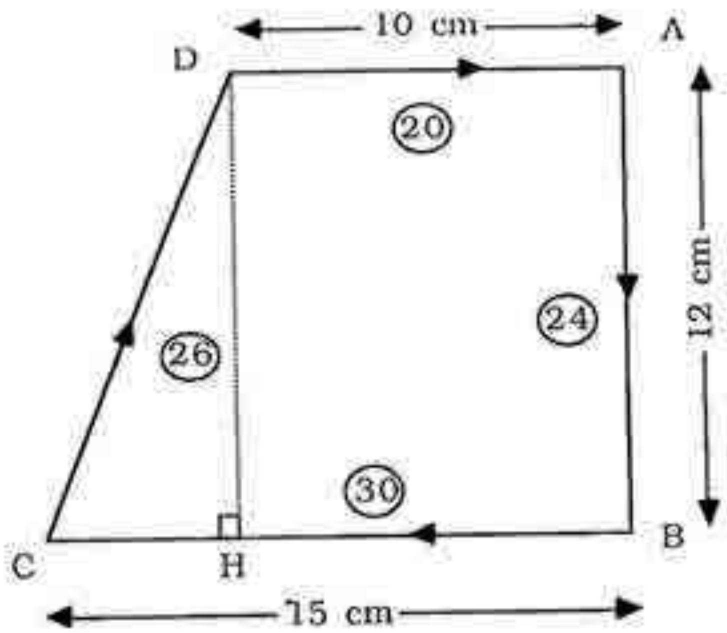


Fig. (89)

$\because$ CH = 5 cm $\qquad\therefore$ CD = 13 cm

$\because$ The forces taken the same way round

$\because$ $\frac{20}{10} = 2,\ \frac{24}{12} = 2,\ \frac{30}{15} = 2,\ \frac{26}{13} = 2$

$\therefore \frac{F_1}{DA} = \frac{F_2}{AB} = \frac{F_3}{BC} = \frac{F_4}{CD} = 2$

$\therefore$ The system is equivalent to a couple

The norm of moment of the couple = twice the area of trapezium divided by the drawing scale.

$\because$ the drawing scale = $\frac{12}{24} = \frac{1}{2}$

The norm of moment of couple= $2 \times \frac{10 + 15}{2} \times 12 \div \frac{1}{2}$ =600 N.cm.

Exercises (5 - 2)

1) ABCD is a square, the length of whose side is 20 cm. Forces of magnitudes, 3, 5, 3, 5, kg. wt act along $\overrightarrow{BA}$, $\overrightarrow{BC}$, $\overrightarrow{DC}$, $\overrightarrow{DA}$ respectively. Two other forces each of magnitude $4\sqrt{2}$ kg.wt act at the two vertices A, C in the directions $\overrightarrow{BD}$, $\overrightarrow{DB}$ respectively. Find the moment of the resultant couple.

2) ABCD is a square of side lenght L. $P \in \overline{AB}$, $Q \in \overline{DC}$, such that $PB = QD = \frac{1}{\sqrt{3}}$ L. Two forces each of magnitude 100 newtons act along $\overrightarrow{AD}$, $\overrightarrow{CB}$. Two other forces each of magnitude 150 newtons ace along $\overrightarrow{PC}$, $\overrightarrow{QA}$. Find the moment of the resultant couple.

3) ABCD is a rectangle, in which AB = 10 cm, CB = 12 cm. X is the midpoint of $\overline{CD}$, Y is the midpoint of $\overline{CD}$. Forces of magnitudes 180, 200, 180, 200, 260, 260 gm. wt along $\overrightarrow{AB}$, $\overrightarrow{CB}$, $\overrightarrow{CD}$, $\overrightarrow{AD}$, $\overrightarrow{AY}$, $\overrightarrow{CX}$ respectively. Find the moment of the resulting couple.

4) ABCDEF is a regular hexagon, of side length 15 cm. Forces of magnitudes 40, 50, 30, 40, 50, 30 newtons act along $\overrightarrow{AB}$, $\overrightarrow{CB}$, $\overrightarrow{CD}$, $\overrightarrow{DE}$, $\overrightarrow{FE}$, $\overrightarrow{FA}$ respectively. Find the moment of the resulting couple of the system.

5) ABCDEF is a regular hexagon of side length 10 cm. Forces of magnitudes 7, 4, 7, 4 gm. wt act along $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{ED}$, $\overrightarrow{EF}$

respectively. Two other forces each of magnitude P gm. wt act along $\overrightarrow{CD}$, $\overrightarrow{FA}$. Find the value of P if the system is in equilibrium.

6) ABCD is a parallelogram, in which AB = 16 cm, BC = 20 cm, m ($\angle$ ABC) = 120°. Forces of magnitudes 3, 5, 3, 5, kg. wt act along $\overrightarrow{AB}$, $\overrightarrow{CB}$, $\overrightarrow{CD}$, $\overrightarrow{AD}$ respectively. Find the magnitude of the resulting couple.

7) ABCD is a square, of side length 30 cm. Forces of magnitudes 4, 5, 4, 5, newtons act along $\overrightarrow{AB}$, $\overrightarrow{CB}$, $\overrightarrow{CD}$, $\overrightarrow{AD}$ respectively. Two other forces each of magnitude $3\sqrt{2}$ newtons act at the two veretices A, C in the directions $\overrightarrow{BD}$, $\overrightarrow{DB}$ respectively. Find:

1st: the couple equivalent to the system

2nd: the magnitude and direction of the each of two forces acting at B, D and which are parallel to $\overleftrightarrow{AC}$ such that the system will be in equilibrium.

8) ABCD is a rectangle, in which AB = 30 cm, BC = 40 cm. Forces of magnitudes 15, 30, 15, 30 newtons act along $\overrightarrow{BA}$, $\overrightarrow{BC}$, $\overrightarrow{DC}$, $\overrightarrow{DA}$ respectively. Prove that the system tends to a couple and find its moment. Find also two forces acting at A, C perpendicular to $\overline{AC}$ in the position of equilibrium.

9) ABCD is a parallelogram, in which AB = 6 cm, BC = 8 cm, m ($\angle$A) = 60°. Forces of magnitudes 6, 9, 6, 9 gm. wt act along $\overrightarrow{AB}$, $\overrightarrow{CB}$, $\overrightarrow{CD}$, $\overrightarrow{AD}$, respectively. Prove that the system tends to a couple and find its moment.

10) Three forces are completely represented by the sides of an equilateral triangle ABC, taken the same way round, and with a drawing scale 1 cm to 2 gm. wt. If the length of the side of the triangle is 30 cm, find the magnitude of the resulting couple.

11) Three forces of magnitudes 20, 30, 20 newtons are represented completely by the directed straight segments $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{CA}$, respectively, where AB = AC = 40 cm, BC = 60 cm. Find the magnitude of the resulting couple.

12) ABCD is a trapizuim in which $\overline{AD}$ // $\overline{BC}$, m ($\angle$ABC) = 90°, AB = 9 cm, BC = 24 cm, AD = 12 cm. Forces of magnitudes 48, 18, 24, 30 newtons acted along $\overrightarrow{CB}$, $\overrightarrow{BA}$, $\overrightarrow{AD}$, and $\overrightarrow{DC}$ respectively. Prove that the system of these forces is equivalent to a couple and find the norm of its moment.

13) ABCD is a quadrilateral in which AB = 8 cm, BC = 6 cm, CD = DA = 13 cm, m ($\angle$ABC) = 90°. Forces of magnitudes 4, 3, 6.5 , 6.5 newtons acted along $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{CD}$, and $\overrightarrow{DA}$ respectively. Prove that the system of these forces is equivalent to a couple and find the norm of its moment. If two forces of magnitudes F, F acted at the two points B and D along $\overrightarrow{CA}$ and $\overrightarrow{AC}$ respectively, find F such that the system of forces will be in equilibrium.

14) ABCD is a rectangle in which AB = 9 cm, BC = 24 cm. H, E are the midpoints of $\overline{BC}$ and $\overline{AD}$ respectively. Forces of magnitudes

27, 72, 45, 36 newtons act along $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{CE}$, $\overrightarrow{EA}$ respectively. Prove that the system tends to a couple and find the norm of its moment. Find two forces if act along $\overrightarrow{HA}$, and $\overrightarrow{EC}$, the system will be in equilibrium.

Part 2

Dynamics

Chapter One
Newton's laws of Motion

Preface :

In this chapter we will deal with: Some basic concepts of dynamics such as force and mass.

Objectives:

By the end of teaching this chapter, the student should be able to:

(1) Recognize the concept of mass and momentum and their units.

(2) Recognize Newton's laws of Motion.

(3) Apply Newton's laws of Motion in different situation.

Topics:

(1) Mass and momentum, their units.

(2) Newton's first law and application.

(3) Newton's second law of motion.

(4) Newton's third law of motion.

(5) Simple applications on Newton's laws of motion.

Newton's laws of Motion, and Simple Applications

A long time ago, the study of the motion of bodies was concerned with two main subjects namely, the motion of celestial bodies (in particular the motion of planets about the sun), and the motin of earth bodies, i.e. those bodies which on the earth's surface or near to it.

At the beginning, scientists and philosophers Aristotle on top of them, thought that these two kinds of motion are mainly different in their basis. This belief has not changed until the second half of the seventeenth century when the British scientist Issac Newton (1642 - 1727) discovered that these two kinds of motion are two faces of the same coin, and that both can be classified under the same title which is "Motion of bodies" in general.

This unification is in fact considered the most important achievement of Newton.

By discovering the general law of attraction, Issac Newton is considered the main founder of modern mechanics science, but the work of other scientists such as Copernick, Kepler, Galilio, proved the way for Newton's achievement in this matter.

The Polish scientist, Nicola Copernick's instructions (1473 - 1543) stated that the earth was a sphere and that it revolved about its axis and about the Sun and thus overcame the old theories, which considered the earth as fixed and that it was the centre of the universe.

Then came the German scientist Johann Kepler (1571 - 1630) who laid down the mathematical basis which govern the motion of planets about the sun, and corrected the ideas about the orbits of the celestial bodies, by showing that the orbits of planets about the sun are ellipses and not circles.

The Italian scientist, Galilio Gallily (1564 - 1642) is considered in fact the founder of the science of motion. Galilio made many experiments on falling and on projected bodies, and also on bodies moving on horizontal surfaces and discovered some of the important properties of its motion. Thus through his experiments on falling bodies, Galilio discovered that in the case of neglecting air resistance all falling bodies move with the same uniform acceleration. He also proved that a projected moves in a trajectory in the form of parabola contrary to what was believed at that time. In his experiments on the motion of bodies on horizontal surfaces, he obtained an important result, which states that the bodies which move on horizontal surfaces without resistance continue their motion with uniform velocity. It is thought that Galilio had reached through his experiments Newton's first and second laws of motion, but he was not able to formulate them clearly then.

Issac Newton collected his researches in a book called "Mathematical principles to natural Philosophy" known as (Principia) i.e. principles in Latin. The first edition of this book appeared in 1686. Principia is considered one of the important scientific books that appeared in modern times, if not the most important of all, and in it Newton formulated his celebrated three laws.

These laws contain as it will be mentioned later, some of the main concepts in mechanics, as the concept of force, and the concept of mass. These concepts have been subject to many discussions among scientists, due to the fact that they are not quite clear.

There is no doubt that we can feel the effect of force from our daily experience. If you watch a horse pulling a cart along a horizontal road, you will find that the cart will move whenever the horse pulls it, and it stops when the horse ceases to pull, and if you push a piece of wood placed on a horizontal table, you will find that it moves under the effect of your pushing, and if you stop pushing it will rest on the table. These two examples imply that bodies act on each other when they touch (we say that bodies act on each other by force), in the first example the horse acts on the cart, and your hand acts on the piece of wood in the second example.

But it is not always so clear. Aristotle asked : what makes a thrown stone move in the air after leaving your hand ? Has your hand transferred to it a quantity of the force which makes it move after projecting it ?

Also, if you leave a body in the air, it will fall towards the earth, what makes it move in this way ?

These questions remained the subject of interest and curiosity of scientists for hundreds of years until Issac Newton discovered his celebrated law of attraction by which he explained the motion of planets about the sun and the motion of falling bodies and projectiles. This law has shown us for the first time that force can produce an effect at a distance, since bodies attract each other without being in touch, and as an example, earth attracts with a force called, the Force of Weight.

But the concept of mass was not so clear as the concept of force. The conept of weight was known to mankind through the needs to make commercial transactions. Then came the invention of the balance to compare different weights.

The concept of mass remained ambiguous and mixed with the concept of weight for a long time even some of the distinguished scientists such as Galilio, Decart, Leibnitz.

The distinction between the two concepts became clear when it was discovered that the weight of the same body may differ from a place to another on the Earth's surface. The mass of a body was defined at the beginning as the "Magnitude of matter it contains". Issac Newton said in one of his writings "I performed a number of precise experiments, and I found each time that the magnitude of matter's body contains is poportional to its weight".

And thus mass appeared as a concept independent of weight, although there is a proportionality between them, since a body of greater weight must have greater mass.

We notice that this static definition of mass does not permit determining the mass of bodies, but the comparison between them, by comparing their weights, as an example we say that the mass of this body is three times the mass of that body, since the weight of the first is three times the mass weight of the second. It is possible to give a dynamic definition (mobile) to the mass by studying the motion of bodies under the action of a given force or the effect of another body and so it is said that the mass of a body is a measure for how far a body can resist the forces whih tends to change its state, or two bodies are left to move under their mutual attraction, and each acquired an acceleration equal in magnitude to the other, then these two bodies are equal in mass.

We should admit that these definitions in their present form are considered ambiguous in a way, or unpractical if we want to use them to find the mass of a body. We are going to discuss this matter later on in this book.

MASS

Consider the following postulate :

Each body is characterized by a self-property called mass, which is a positive scalar quantity, directly proportional with the weight of this body, on condition that all weights are measured in the same place on the earth's surface.

Usually the mass is denoted by m.

From the above postulate we have the important property of the mass, which is the property of summation :

"The mass of a body is equal to the sum of the masses of its components".

This property is shown in fig. (1)

body (1) m_1	+	body (2) m_2	=	body (1 + 2) $m_1 + m_2$

Fig. (1) : masses summation porperty

And as it is clear, the previous postulate means that the mass of a body is a measure of what it contains of matter. But does this postulate enable us to define mass of bodies ? The answer is no. All we can do is a comparison between the masses of bodies. To find the mass it is necessary to define the "unit of mass", and that is what we are going to explain in the next article.

Units of measure of mass :

We have seen that to find the mass of a body it is necessary to define the unit of mass, which is the mass of unit magnitude, and with which we will compare the other masses. Scientists have agreed that the unit of mass is the "kilogram" (kg.), which is the mass of a cylindrical body made of the two metals, Platinium and Iridium kept in the museum of the internatinal office for scales and measurements in the town Seevre, France.

The kilogram is equal to the mass of one litre of distilled water at 4 centigrade. If we consider the metric system, we will find that it contains, in addition to the kilogram, a numerous number of units of measure of the mass, such as the Ton, Gram, Decigram, Centigram, Milligram, Microgram.

Here are the rules for transferring some of the units of mass.

1 ton	=	1000 kilogram
1 kilogram	=	1000 gram
1 gram	=	1000 milligram.

It is clear that the gram and its divisions are used in measuring relatively small masses (for example in medicine industry), while the ton is used in measuring relatively big masses (for example in heavy industry or agriultural crops).

It should be mentioned that the mass of a body may change from instant to instant. There are many examples : the mass of a rocket diminishes due to the ejection of burning gasses, also the mass of a raindrop increases during falling down due to accumulation of moisture on its surface.

Example (1) :

A rocket originally of mass 15 tons, throws off fuel at a constant rate of 200 kg per secod. Find the mass of the rocket after 30 seconds from the instant of firing.

Solution :

Since the rocket throws fuel at a constant rate, therefore mass of fuel thrown out = rate × time

$= \; 200 \times 30 = 6000 \text{ kg} = \; 6 \text{ tons}$

$\therefore$ mass of rocket after 30 seconds is

$m = 15 - 6 = 9 \text{ tons}$

Example (2) :

A falling raindrop is of mass 0.1 gram at a certain instant. If vapour accumulates on its surface during its fall at the rate of 2 milligram per second, find the mass of the raindrop after 1 minute from this instant.

Solution :

The acquired mass = rate × time

$= 2 \times 60 = 120$ milligrams

$= 0.12$ gram

$\therefore$ the final mass $= 0.1 + 0.21 = 0.22$ gram.

Momentum

The mass of a moving particle and its velocity vector form an important property of the motion characteristics, called **Momentum.** The Italian scientist Galillio was the first to notice the importance of the product of the weight of a body and its velocity when studying the motion, but it seems that the French scientist Decart was the first to use the expression of momentum, and defined it as the product of the mass and the

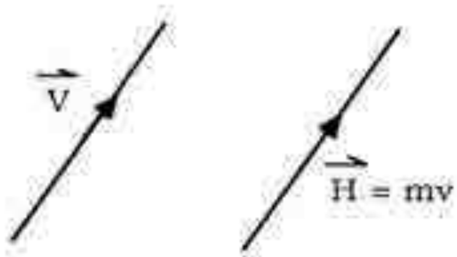


Fig. (2)

velocity. The extreme importance of this last definition appeared through the work of Newton and Huggins.

Definition :

The momentum vector of a particle, denoted by $\vec{H}$ is defined as the product of the mass of the particle and its velocity vector. Fig. (2).

$$\vec{H} = m \vec{v} \qquad (1)$$

From this definition it is clear that the momentum of a particle at a certain instant is a vector in the same direction as the instantaneous velocity vector at this instant, and the momentum vector of a particle changes from instant to instant in magnitude and direction according to the change of the instantaneous velocity vector. We can notice the effect of momentum in many of the surrounding appearences, as an example if you put a grain of sand on your hand you will not feel its effect, but this particle of tiny mass can scratch the glass of a rapidly moving car in a sandy storm. The reason is that the grain of sand had acquired momentum relative to the car (equal to the product of its mass into its velocity relative to the car) and the magnitude of its momentum vector became very large to a certain extent due to the magnitude of its relative velocity vector.

Also if you thrown a big stone on a solid wall it will no penetrate the wall, but if you fire a bullet on the same wall it will be imbedded in the wall, the difference here is that the velocity of the bullet is much greater than that of the stone although the mass of the stone may be greater than that of the bullet.

In the case of rectilinear motion, the two vectors $\overrightarrow{H}$, $\vec{v}$ will be parallel to the straight line on which the motion occurs, and thus it is possible to express both of them in terms of their algebraic measures relative to a unit vector parallel to this line.

$$\overrightarrow{H} = H\ \hat{c} \qquad\qquad \vec{v} = V\ \hat{c}$$

Substituting in (1), and eliminating $\hat{c}$ from both sides, we get.

$$H = m\ v \qquad\qquad (2)$$

i.e. The algebraic measure to the momentum vector is equal to the product of the mass times of the algebraic measure of the velocity vector.

Units of measure of the magnitude of momentum :

As the magnitude of the velocity vector if measured by units of magnitude of velocity, the magnitude of momentum vecotr is measured by units of magnitude of momentum.

Unit of measure of magnitude of momentum

= unit of measure of mass × unit of measure of magnitude of velocity.

As an example, we can measure magnitude of momentum by units of

$$\text{gram} \times \frac{\text{centimeter}}{\text{second}} \quad (\text{gm.cm./sec.})$$

or by units of

$$\text{Kilogram} \times \frac{\text{kilometer}}{\text{hour}} \quad (\text{kg.km/h})$$

N.B. Sometimes when we are not concerned about the direction we will use the expression momentum to mean the magnitude of momentum.

Example (3) :

Find the momentum of car whose mass is 1.5 tons moving with a speed of 80 km/h.

Solution :

Momentum = mass × speed

= 1.5 x 80 = 120 ton. km/h.

(Speed = magnitude of velocity)

Example (4) :

Compare the momentum of a train of mass 12 tons moving with a speed of 0.3 km/h and the momentum of the shot of a gun of mass 2.5 kg moving with a speed of 400 m/sec.

Solution :

Momentum of the train = 12 × 0.3 ton.km./h

= 3.6 ton.km/h

$$= 3.6 \times 10^6 \times \frac{10^5}{3600}$$

$$= 10^8 \text{ gm.cm./sec.}$$

Momentum of the shot of the gun $= 2.5 \times 400$ kg.m./sec.

$$= 10^3 \text{ kg.m/sec.}$$

$$= 10^3 \times 10^3 \times 10^2 \text{ gm.cm/sec.}$$

$$= 10^8 \text{ gm.cm/sec.}$$

i.e. the train and the shot have the same momentum.

Example (5) :

A rubber ball of mass 400 gm is let to fall on a horizontal gound, its velocity when it impinges on the ground is 100 cm/sec. then it rebounds from the ground with a veloity of 60 cm/sec. Find the change in momentum as a result of impact.

Solution :

Let $\hat{j}$ be a unit vector diected vertically downwards, fig. (3).

The velocity vector of the ball just before impact has the same diection as $\hat{j}$

$$\overrightarrow{v_1} = 100\,\hat{j}$$

Momentum vector of ball just before impact

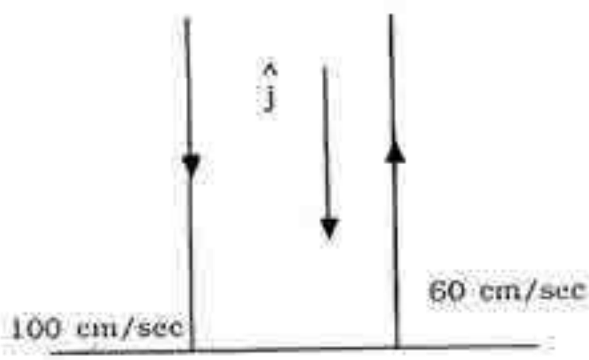


Fig. (3)

$$\overrightarrow{H_1} = m\,\overrightarrow{v_1} = 400 \times 100\,\hat{j} = 40000\,\hat{j}$$

Velocity vector of ball just after impact (direction of velocity vector opposite to that of $\hat{j}$)

$$\overrightarrow{v_2} = -60\ \hat{j}$$

Momentum vector of all just after impact

$$\overrightarrow{H_2} = m\ \overrightarrow{v_2} = 400 \times (-60\hat{j}) = -24000\hat{j}$$

Change in momentum due to impact $= \overrightarrow{H_2} - \overrightarrow{H_1}$

$$= (-24000\hat{j}) - (40000\hat{j})$$

$$= -64000\hat{j}$$

Magnitude of change in momentum

$$= \| H_2 - H_1 \| = 64000 \text{ gm.cm/sec.}$$

Exercise :

The student has to solve this problem when $\hat{j}$ is directed vertically upwards.

Example (6) :

A car moves along a straight road with a velocity of magnitude 100 km/h, and there is a sandy storm in an opposite direction to that of the car with a velocity of 80 km/h. If the mass of a grain of sand is 10 milligrams, find the momentum of the grain of sand relative to the car, measured in units of gm.cm/sec.

Solution :

Consider a unit vector $\hat{i}$ in the direction of motion of the car, fig. (4). Velocity vector of car $\overrightarrow{v_1} = 100\ \hat{i}$. Velocity vector of the grain of sand $\overrightarrow{v_2} = -80\hat{i}$. Since it is required to find the momentum of the grain of sand relative to the car, we have to find the velocity of the grain relative to the car.

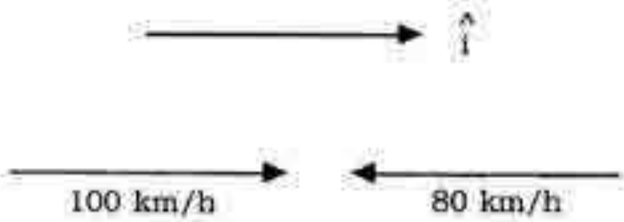


Fig. (4)

Let this velocity be $\vec{v}$. From the rules of relative velocity, we find

$$\vec{v} = \overrightarrow{v_2} - \overrightarrow{v_1} = (-80\hat{i}) - (100\hat{i})$$
$$= -180\hat{i}$$

which means that a viewer inside the car see the grain of sand as if it were moving with a velocity of 180 km/h in a diection opposite to that of the car.

Momentum of grain of sand relative to the car

$$\vec{H} = m\vec{v} = 10 \times (-180\hat{i}) = -1800\hat{i}$$

$\therefore \|H\| = 1800$ milligram. km/h.

$$= 1800 \times 10^{-3} \times \frac{10^5}{3600} \text{ gm.cm/sec.}$$
$$= 0.5 \times 10^2 = 50 \text{ gm.cm/sec.}$$

Example (7) :

A particle is let to fall from the top of a tower. Calculate its momentum at any time instant and prove that its rate of change is constant.

Solution :

From the motion laws under the acceleration due to the gravity, the velocity vector of a particle at any time t is

$$\vec{v} = \vec{v_\circ} + t\,g\,\hat{j}$$

$$= \vec{O} + t\,g\,\hat{j} = t\,g\,\hat{j}$$

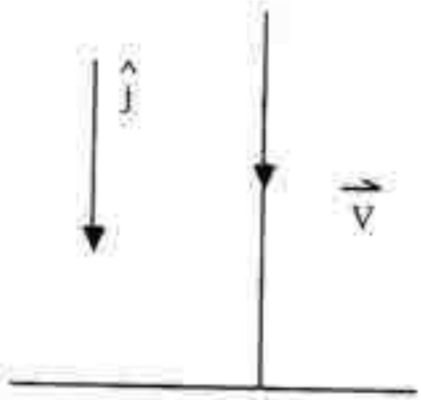


Fig. (5)

where g is the acceleration due to gavity, $\hat{j}$ is a unit vector shown in fig. (5)

$$\therefore\ \vec{H} = m\,\vec{v} = t\,m\,g\,\hat{j}$$

Rate of change of $\vec{H}$ with respect to time

$$\frac{d\vec{H}}{dt} = m\,g\,\hat{j}$$

which is a constant vector

Exercises (1 -1)

1) A rocket of mass 3 tons throws out fuel at a constant rate of 100 kg per second. If the mass of the rocket when empty of fuel is 1 ton, find when all the fuel is thrown out.

2) A sphere of mass 1 kg moves in air saturated with dust. If the rate of accumulation of dust on its surface is 20 gm/min. How long will it take until the mass of the sphere be equal to 1.5 kg?

3) Find the momentum of a car of mass 1800 kg moving with a speed of 100 km/h giving you answer in units of gm.m./sec.

4) A rubber ball of mass 40 gm. is pojected on a smooth horizontal ground, it impinges with a barrier with a velocity 80 cm/sec., then it rebounds in the opposite direction with a velocity 40 cm./sec. Find the magnitude of the change in momentum due to impact.

5) A rubber ball of mass 100 gm is let to fall on a horizontal ground, it impinges with the ground with a velocity 400 cm./sec., then it rebounds to a height of 50 cm before it comes to instantaneous rest. Find the magnitude of the change in momentum just before and after impact.

6) A rubber ball of mass 50 gm is let to fall from a height of 4.9 m on a hoizontal ground, it impinges with it and then rebounds to a

height of 2.5 m before it comes to instantaneous rest. Calculate the magnitude of the change of its momentum just before and after impact.

7) A rubber ball of mass 100 gm is let to fall fom a height of 40 cm. on a hoizontal ground. If the ball rebounds to one quarter of the height fom which it falls after each impact, find the magnitude of the change in its momentum, just before and after the second impact, measured in units of gm. cm/sec.

8) A bullet of mass 50 gm is fired with velocity 810 m/sec. towards a wooden body of mass 4 kg which is at rest. If the bullet is imbedded in it, and the system moves after that with a certain velocity, find this velocity given that the momentum of the system did not change due to impact.

9) An anti-tanks gun fires a shell of mass 1 kg with velocity 300 m/sec. towards a tank moving towards the gun with velocity 60 km/h, and it hits it. Find the absolute value of the momentum of the shell, and also the magnitude of its momentum relative to the tank. Compare the two results.

10) A particle is projected vertically upwards with a velocity v_o. Write down the law which gives its velocity in terms of the time, and hence deduce that the rate of change of the momentum with respect to the time is a constant vector.

NEWTON'S LAWS

Newton's First Law :

"Every body perseveres in its state of rest or of moving uniformly except in so far as it is made to change that state by external effect.

The student is reminded that a uniform motion is a motion with constant velocity in magnitude and direction".

Discussion of the Law :

1) The law assumes the existence of an effect, called "the force" which if it acts on a body at rest or moving uniformly, it changes its state.

And thus if you see that a body which is at rest starts to move, then it must have been under the effect of a force, and te body moves on a curved path as in fig.(6), then it is under the effect of a force. Also if a body moves in a straight line with a variable velocity in magnitude or direction or in both, we can deduce that there is a force acting on it.

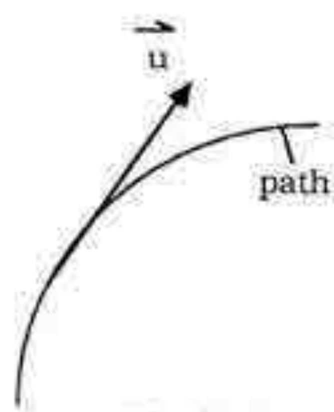


Fig. (6)

2) In the formulation of the law, the force means the resultant of all forces acting on the ody.

3) The law consides the two states of rest and uniform motion in an equivalent situation, as both represent the natural state of the body when the resultant of the acting forces is equal to zero.

4) The law shows that a body which is at rest or moving uniformly (i.e. when it is in its natural state) cannot change its state by itself, but a force must act on it to change its state, and that is why the first law is called the Law of Inertia.

5) The first law agrees with the results deduced by Galilio through his experiments on balls moving on hoizontal surfaces, for he noticed that when the resistance of the surface to motion became smaller, the body approached the state of unifom motion.

It is useful to know that Galilio had formulated a law similar to Newton's first law, which states that "If a body is moving on the earth's surface without any resistance acting on it, it will move with a constant velocity on a great circle of the earth".

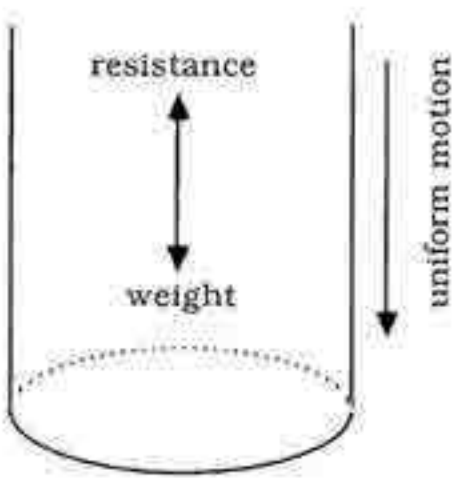
6) Although the fomulation of Newton's first law is so simple, yet this law contains deeper significance than what its formulation implies.

Fore more explanation,we remind the student that the concept of motion is a relative concept, and that what a viewer considers as movable, may appear at rest to another viewer.

Therefore, we expect the existence of some limits of validity to Newton's first law, that is to say, that this law will not be true to all viewers studying motion of bodies, but to some of them. From our experience, we can see that a viewer who observes the motion from a position on the earth's surface can use Newton's first law, and obtain results that agree with reality, on condition that the motion which he observes is limited in space and time to a sufficient degree, that is to say the body has not covered large distance, or moved for a long time than is necessary. In other cases, the rotation of earth about its axis and about the sun, makes it very dangerous to apply the first law, and may lead to wrong results.

Example (1) :

To prove that a liquid resists the motion of bodies in it. If a metal ball is let to fall in a long vertical tube full of a liquid (such as oil), then we notice that its motion will be accelerated at the beginning then it becomes uniform, fig. (8).



Fi. (8)

We can be sure of the uniform motion experimentally by noticing that the ball descends equal distances in equal intervals of time.

By applying Newton's first law on the mentioned uniform motion, we conclude that the resultant of the acting forces must vanish. Since the ball is descending under its weight which is directed vertically downwards, we deduce the existence of a second force that balances the weight, i.e. it is directed vertically upwards, and equal in magnitude to the weight of the ball, acting on the ball. We know that this force is a result of the liquid's particles touching the surface of the ball, and is called "liquid's resisting force" to the motion of a body in it. This force is responsible for the viscosity phenomena.

Example (2) :

Uniform motion of a body on a horizontal ground. Consider a body moving uniformly on a horizontal ground under the action of a force $\vec{F}$ in direction inclined at an angle of measure θ to the horizontal upwards as in fig. (9).

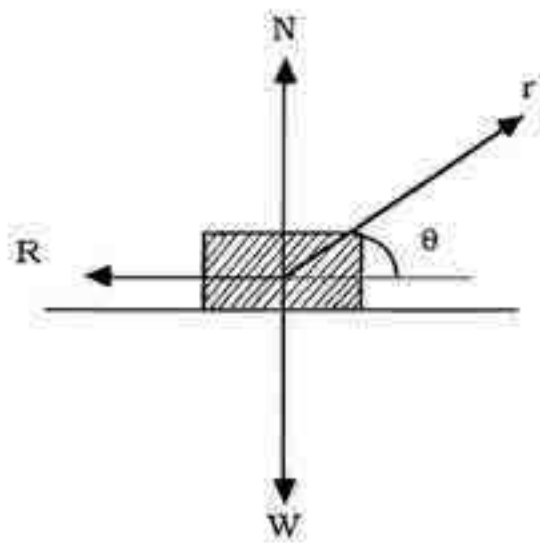


Fig. (9)

Since the motion is uniform, the algebraic sum of the components of the acting forces on the body must vanish. Since the force of weight is vertical, it has no component in the horizontal direction. We therefore deduce the existence of a force acting on the body in an opposite direction to that of motion to balance the component $F \cos\theta$ of the force $\vec{F}$. This force is called "The force of resistance of the earth to the motion of the body on it" and is due to the body touching the ground during its motion. Denoting this force by R, therefore $R = F \cos\theta$. Also since there is no motion of the body in a vertical direction, the algebraic sum of the components of the acting forces on the body in that direction must vanish. Therefore there must exist a force acting on the body vertically upwards to balance the force of weight. This force is called "the normal reaction of the ground". This force is also due to the contact between the body and the earth. We will denote this force by N. From the figure,

$$\therefore\ N + F \sin\theta = W.$$

Example (3) :

Uniform motion of a car or a train on a horizontal road. In this case, the deriving force is that of the motor and its direction is horizontal as in fig(10).

If we put $\theta = 0^\circ$ in the previous example, we get

$$R = F$$

$$N = W$$

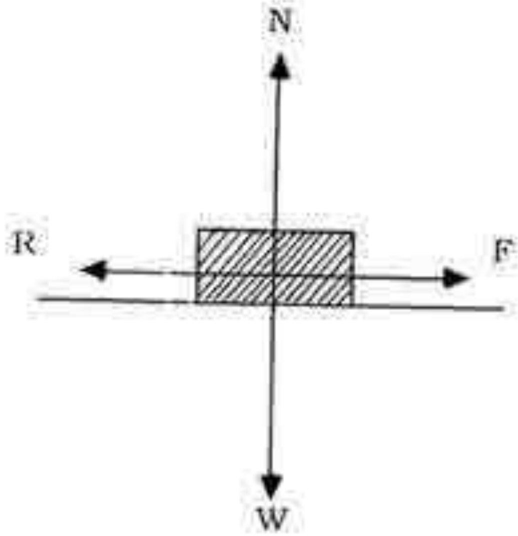


Fig. (10)

But if the moving body is an aeroplane flying uniformly at a constant height, then the force which is directed vertically upwards and acting on the plane is the force of lift, which acts mainly on the plane's wings, and is due to the difference between the air velocity relative to the plane, below and above the wings.

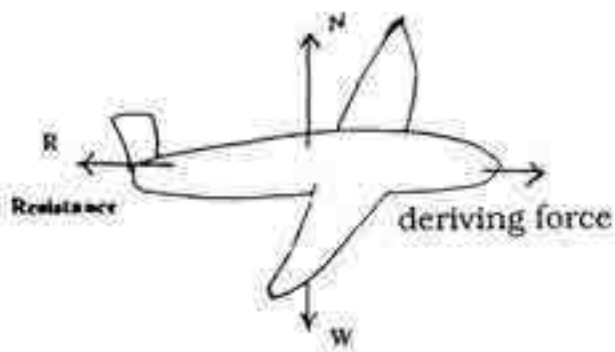


Fig. (11)

Example (4) :

A car of mass 3 tons moves on a straight horizontal road, the force of resistance due to friction is directly proportional to the magnitude of the car's velocity. If the maximum pulling force of the motor is 500 kg. wt, and the force of resistance is 50 kg. wt for each ton of the mass of the car, when the car's velocity was 30 km/h, find the maximum velocity of the car on the road.

Solution :

The concept of maximum velocity includes two important notices :

a) The car is moving with a constant speed, which is the magnitude of the maximum velocity.

b) The motor its working at its maximum force.

The force acting on the car are :

1) The deriving force of the motor acting horizontally in the direction of motion.

2) The weight force acting vertically downwards.

3) The resultant reaction, which can be resolved into two components, one of

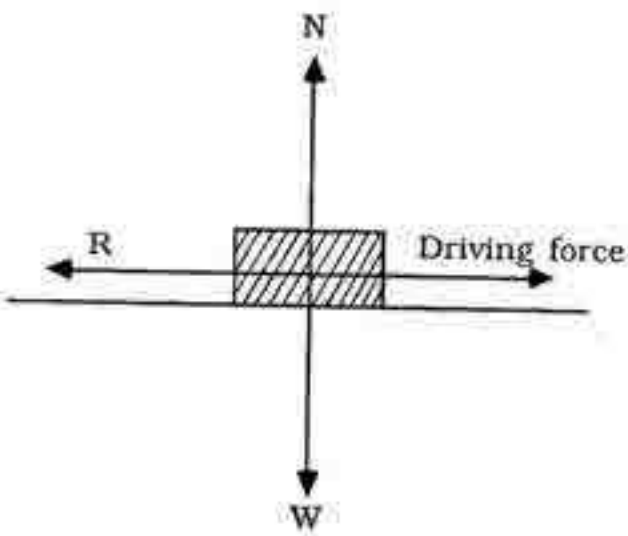


Fig. (12)

them is the force of resistance (friction) acting in a direction opposite to that of motion,the second is the force of normal reaction acting vertically upwards, as in fig. (12).

Since the motion is in the horizontal direction, then the normal reaction is equal in magnitude, and in an opposite direction to the weight. Therefore their resultant vanishes and thus we can disregard these two forces if it is not required to find the magnitude of the normal reaction forces.

Let R be the resistance force, v the speed.

$$\therefore \quad R = kv \qquad (1)$$

where k is the constant of proportionality, and which we can determine (if necessary) from the knowledge of the resistance force when the velocity is 30 km/h. When the velocity is 30 km/h the resistence force is

$$R = 50 \times 3 = 150 \text{ kg. wt.}$$

Substituting in (1) $\qquad 150 = k \times 30 \qquad (2)$

Let v_1 be the car's maximum velocity when the resultant of the acting forces on the car vanishes, and thus the resistance force is equal and opposite in direction to the deriving force of the car.

$$\therefore \quad R = 500 \text{ kg. wt.}$$

Substituting this value in equation (1), putting

$$v = v_1$$

$$\therefore \quad 500 = k\ v_1 \qquad (3)$$

dividing (3) by (2)

$$\therefore \frac{500}{150} = \frac{k\,v_1}{k \times 30}$$

$$= \frac{v_1}{30}$$

$$\therefore v_1 = \frac{500}{150} \times 30 = 100 \text{ km/h.}$$

Example (5) :

A body moves under the effect of two forces $\overrightarrow{F_1} = \hat{i}$ and $\overrightarrow{F_2} = -\sqrt{3}\ \hat{j}$, where $\hat{i}$ and $\hat{j}$ are orthogonal unit vectors. Find the third force $\overrightarrow{F_3}$ which if it acts on the body, it will move uniformly, find also the norm and the direction of $\overrightarrow{F_3}$.

Solution :

Since the body moves uniformly.

$\therefore \overrightarrow{F_1} + \overrightarrow{F_2} + \overrightarrow{F_3} = \overrightarrow{0}$

$\therefore \overrightarrow{F_3} = -(\overrightarrow{F_1} + \overrightarrow{F_2})$

$= -(\hat{i} - \sqrt{3}\ \hat{j})$

$= -\hat{i} + \sqrt{3}\ \hat{j}$

$\therefore \|\overrightarrow{F_3}\| = \sqrt{1+3} = 2$ units,

$\tan\theta = \dfrac{\sqrt{3}}{-1} = -\sqrt{3}$ $\quad\therefore \theta = 120°$, where θ is the angle between the line of action of $\overrightarrow{F_3}$ and the unit vector $\hat{i}$.

Exercises (1 - 2)

1) Figure (13) shows the forces acting on some bodies. Which of these bodies can be considered in a state of rest or moving uniformly ?

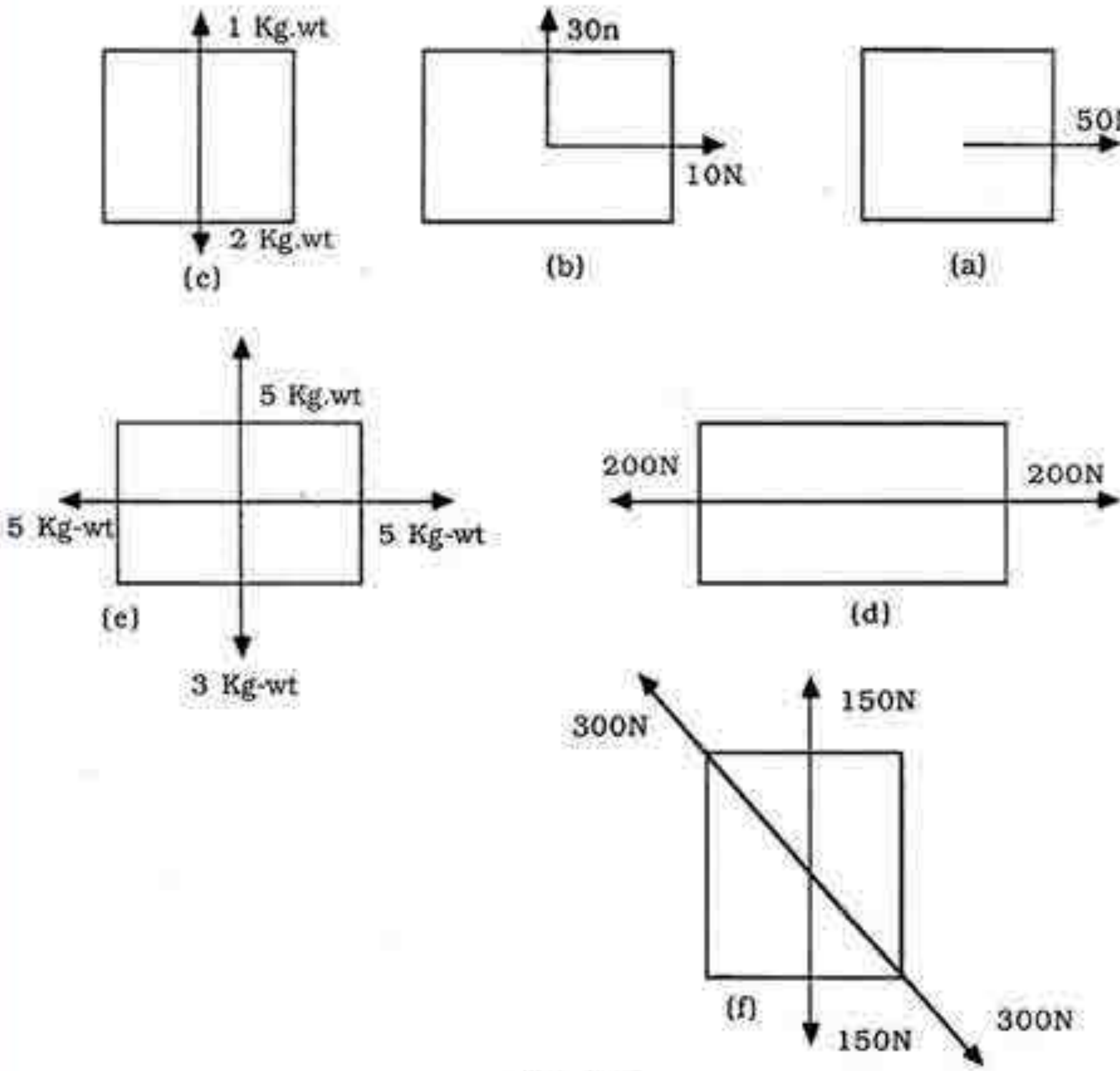


Fig. (13)

2) A metal ball of weight 150 kg. wt descends vertically in a liquid. It is found that it travels equal distances in equal time intervals. What is the magnitude of the liquid's resistance force to the motion of the ball ?

3) A parachutist descends vertically with uniform velocity. If the total weight of the man and the parachute is 95 kg. wt, find the magnitude of the air resistance force to the parachute.

4) A horse pulls a wooden block on a horizontal ground with a force of magnitude 100 kg. wt in a direction inclined at 30° upwards to the vertical. If the block moves uniformly, find the ground resistance force to its motion.

5) A car of mass 4 tons moves on a straight horizontal road under the action of a resistance directly proportional to the magnitude of the velocity. If the resistance is 8 kg. wt for each ton of the mass of the car when its velocity is 72 km/h, find the mximum velocity, given that the maximum force generated by the motor is 60 kg. wt.

6) A particle of mass m moves under the effect of the two forces

$\overrightarrow{F_1} = 3\,m\,\hat{i}$ $\qquad\qquad$ $\overrightarrow{F_2} = 4\,m\,\hat{j}$.

where $\hat{i}$ & $\hat{j}$ are two perpendicular unit vectors. Find the additional force which if it acts on the particle, it will move uniformly.

7) A man is tied to a parachute descends vertically downwards. Given that the air resistance is directly proportional to the square of the magnitude of the velocity, and is equal to 1/4 the weight of the man and the parachute when the velocity is 15 km/h, find the velocity of descent of the man and parachute, when this velocity becomes uniform.

8) A train of mass 112 tons and the deriving of its engine is 5600 kg. wt. If the resistance to the motion of the train is directly proportional to the square of its velocity, and this resistance is 32 kg. wt for each ton of the mass when its velocity was 60 km/h, calculate the maximum velocity of the train.

9) A body of mass 10 kg is placed on a horizontal plane and is attached by two horizontal strings the angle between them is 120°and the tension is each string is of magnitude 400 gm.wt. If the body moves uniformly on the plane, find the magnitude and the direction of the resistance of the plane.

Newton's Second Law :

Rate of change of momentum with respect to the time is proportional to the acting force and takes place in the direction in which the force is acting

The mathematical expression of the second law is

$$\frac{d}{dt}(m\vec{v}) \propto \vec{F}$$

or

$$\frac{d}{dt}(m\vec{v}) = k\vec{F}$$

where k is a positive constant of proportionality. If the mass of the body is constant during the motion, we can write.

$$\frac{d}{dt}(m\vec{v}) = m\frac{d\vec{v}}{dt} = m\vec{a}$$

where $\vec{a}$ is the acceleration of the body. Thus, Newton's second law takes the form :

$$m\vec{a} = k\vec{F} \qquad (1)$$

If the constant k is known, we can use the above relation to determine the force $\vec{F}$ if the acceleration $\vec{a}$ is known. We shall show how we can find this constant.

Since both the force and acceleration vectors are in the same direction, we can express them in terms of a unit vector $\hat{c}$ in their directions by means of their algebraic measures as follows :

$$\vec{a} = a\hat{c} \qquad\qquad \vec{F} = F\hat{c}$$

In this case the algebraic measures of these vectors are considered positive and equal to its magnitude.

Substituting in (1) we get

$$m\,a\,\hat{c} = k\,F\,\hat{c}$$

eliminating $\hat{c}$ from both sides, we get

$$m\,a = k\,F \qquad (2)$$

To find the value of the constant k we have to translate equation (2) to figures. We know the units of measuring the magnitude of the acceleration, but up till now we haven't defined any units of measuring the magnitude of a force, and that is what we are going to do now.

The newton :

It is a unit for measuring the magnitude of a force, and is defined as the magnitude of a force which if it acts on a body of mass 1 kilogram it acquires an acceleration of magnitude 1 m/sec^2. Fig (14). putting F = 1 newton, m = 1 kg, a = 1 m/sec^2. in eq. (2) we get $1 \times 1 = k \times 1 \qquad \therefore k = 1$

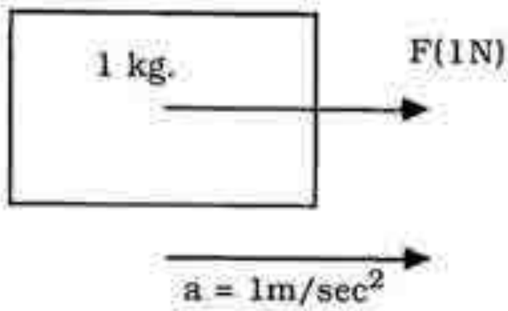


Fig. (14)

and thus Newton's second law takes the vectorial form.

$$\frac{d}{dt}(m\vec{v}) = \vec{F} \qquad (3)$$

and when the mss is constant, we get

$$m\vec{a} = \vec{F} \qquad (4)$$

which means that

"In the case when the mass is constant during its motion, the force vector is equal to the product of the mass of the body times its acceleration vector". But if we use the algebraic measures of the vectors $\vec{a}$, $\vec{F}$, relation (4) takes the form

$$m\,a = F \qquad (5)$$

It is clear that the two quantities a, F have the same sign.

Equation (5) states that

"When the mass of a body is constant during its motion, the algebraic measure of the force vector is equal to the product of the mass of the body times the algebraic measure of its acceleration vector".

Each of the two relations (4), (5) are satisfied if we are obliged to use the above mentioned units in the following way :

$$m \text{ (kg.)} \times a \text{ (m./sec}^2) = F \text{ (newton)} \qquad (6)$$

There are many other units for measuring the magnitude of a force, other than the newton. The most important of these is the kilogram weight (kg. wt.), gram weight (gm. wt.) and the dyne.

Here are the rules for transferring the units of force :

$$1 \text{ kg. wt.} = 9.8 \text{ newtons}$$

$$1 \text{ gm. wt.} = 10^{-3} \text{ kg. wt.}$$

$$1 \text{ dyne} = 10^{-5} \text{ newton} = \frac{1}{980} \text{ gm. wt.}$$

Since relation (5) expresses the direct proportionality between the magnitudes of the force and the acceleration, we can define a kilogram weight as the magnitude of the force which if it acts on a body of mass one kilogram, it acquires an acceleration of magnitude 9.8 meter /sec^2.

Also we can formulate relation (6) in new units by noticing that

$$1 \text{ kg.} = 1000 \text{ gm} , \quad 1 \text{ m/sec}^2 = 100 \text{ cm. /sec}^2$$

$$1 \text{ newton} = 10^5 \text{ dynes,}$$

therefore

$$1000 \text{ m (gm.)} \times 100 \text{ a (cm/sec}^2) = 10^5 \text{ F (dyne)}$$

dividing by 10^5 we get :

$$\text{m (gm)} \times \text{a (cm/sec}^2) = \text{F (dyne)} \qquad (7)$$

This means that in Newton's second law we can measure the magnitude of the mass in grams, the magnitude of the acceleration in units of cm./sec^2, and the magnitude of the force in dynes.

Putting $m = a = 1$ in the above relation we get $F = 1$ therefore "The dyne is the magnitude of a force which if it acts on a body of mass one gram it acquires an acceleration of magnitude 1 cm./sec^2".

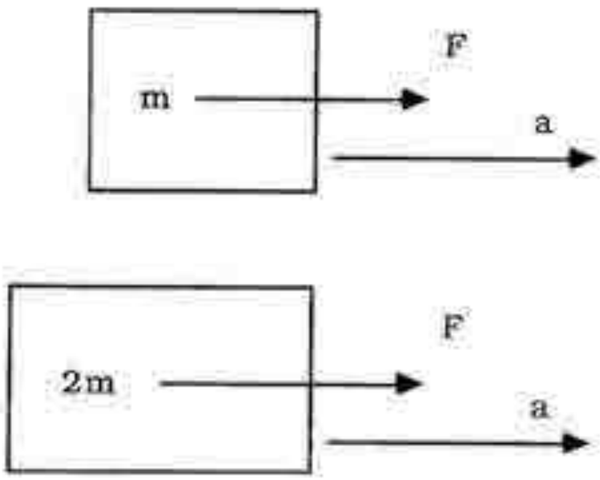
Discossions of the second law :

1) This law expresses Newton's idea which states that the acceleration of a body is a measure to the force acting on it, and thus if we want to describe the force acting on a body, we have to study its acceleration, whatever the nature of this force may be.

2) If we act with the same force $\vec{F}$ on two bodies, one of mass m, and the mass of the second is 2 m fig. (15), using the relation m a = F.

i.e. $a = \frac{F}{m}$

the first body acquires an acceleration (a), while the second body (of greater mass) acquires an acceleration of magnitude (1/2 a), which means that under the action of the same force, the body of greater mass acquires an acceleration of magnitude less than that of the body of smaller mass. Thus the mass appears as an element resisting the effect of the force, and this agrees with the dynamical definition of mass, which states that the mass is a measure to the extent to which a body resists the forces which tend to change its state.



Fi.g (15)

The following diagram shows the linear relation between the force and acceleration for three bodies of different masses, the straight line corresponsing to the body of lesser mass nearer to the acceleration axis. fig. (16).

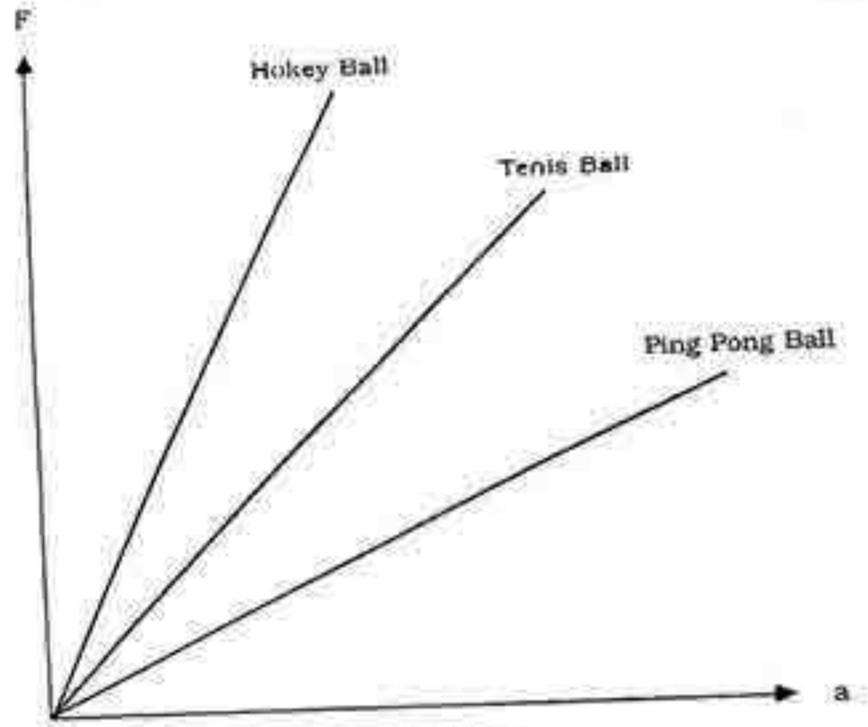


Fig. (16)

3) Neton's second law is considered as a definition of the force in terms of the acceleration, but in the cases in which the force is known from other sources, we can use this law to determine the acceleration of bodies.

4) Galilio has proved through his experiments on falling bodies that when there is no resistance all bodies fall with the same uniform acceleration, for he found from his experiments that the vertical distance which the falling body cut is proportional to the square of the time. i.e.

$$s \propto t^2$$

which agrees with the main law

$$s = 1/2 \; a \, t^2$$

for the motion with uniform acceleration.

Since the acceleration acquired by falling bodies is due to the Earth's attraction to these bodies, the falling acceleration is called "Earth's gravitational acceleration" denoted by (g) fig. (17).

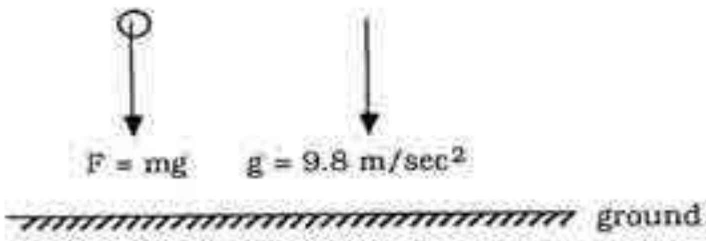


Fig. (17)

Experimentally it was found that the Earth's gravitational acceleration magnitude is given by the approximate value.

$$g = 9.8 \text{ meter/ sec}^2$$

This value increases when we go towards the Poles, where it is equal to 9.83 approximately, while it decreases when we go towards one of the equator, where it is equal to 9.79 approximately.

The force which attracts bodies downwards is the force of weight, whose magnitude will be denoted by (W)

Applying Newton's second law to the motion of falling bodies, i.e. putting.

$$F = W \quad , \quad a = g$$

we find $\quad W \text{ (newton)} = m \text{ (kg.)} \times g \text{ (m./sec}^2\text{)}$

or $\quad W \text{ (newton)} = 9.8\, m \text{ (kg.)} \times 1 \text{ (m./sec}^2\text{)}$

noticing that 1 kg. wt. = 9.8 newtons

$\therefore \quad W \text{ (kg. wt)} = m \text{ (kg)} \times g \text{ (m/sec}^2\text{)}.$

This relation shows that the weight of a body measured by units of kg. wt is numerically equal to the mss of this body measured in units of kg.

For example consider a body of mass 7 kg. i.e. m = 7.

$\therefore$ W (kg. wt.) = 7 × 1 = 7

i.e. the weight of this mass is equal to 7 kg. wt.

The student must not be mixed between the unit of mass (kg) and the unit of force or weight (kg. wt).

5) If the resultant of the forces acting on a certain body vanish i.e. if $\vec{F} = \vec{O}$

according to Newton's second law (3) takes the form

$$\frac{d}{dt}(m\vec{v}) = \vec{0}$$

and thus the momentum vector m $\vec{v}$ must be a constant vector

$\vec{H} = m\vec{v} =$ constant vector.

therefore, the body on which no force acts moves so that its momentum vector is constant, and this means of course, that the motion is in a straight line but its velocity may change from instant to another (because the mass may vary), but if the mass is constant, the above relation gives

$\vec{v}$ = constant vector

i.e. the body in this case moves in a uniform motion, and this is Newton's first law.

Example (1) :

Prove that if a body moves in a straight line fixed in space, then there are two possibilities which may happen : either its motion is uniform, or the resultant of the acting forces on it is parallel to the straight line.

Solution :

Let $\hat{i}$ be a unit vector parallel to the straight line on which the body moves, and $\hat{j}$ a unit vector perpendicular to $\hat{i}$. fig. (18). Let us assume that the motion is not uniform, therefore the acceleration vector is not equal to the zero vector. Since the motion is in a straight line, then the velocity vector is parallel to this line, and thus we can express the velocity vector in terms of its algebraic measure relative to the unit vector $\hat{i}$.

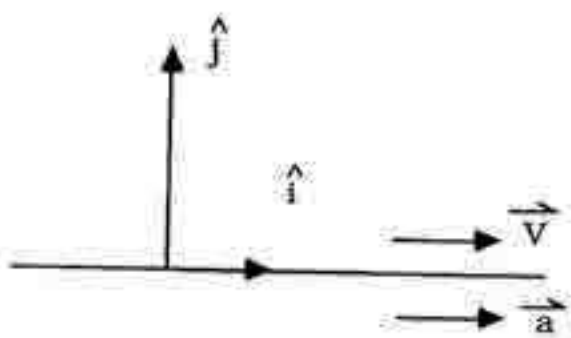


Fig. (18)

$$\vec{v} = v\,\hat{i}$$

the acceleration vector is

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}(v\ \hat{i}) = \frac{dv}{dt}\ \hat{i}$$

i.e. the acceleration vector is also parallel to the straight line, and thus is perpendicular to $\hat{j}$.

$$\therefore\ \vec{a} \odot \hat{j} = 0$$

from Newton's second law

$$\vec{a} = \frac{\vec{F}}{m}$$

$$\therefore\ \vec{a} \odot \hat{j} = \frac{\vec{F}}{m} \odot \hat{j} = 0$$

$$\therefore\ \vec{F} \odot \hat{j} = 0$$

which means that the resultant vector of the acting forces is perpendicular to $\hat{j}$ i.e. it is parallel to the straight line on which the body moves.

N.B.

We are going to study many examples, in which the body is moving on a horizontal table, or on an inclined plane. According to the above example, the resultant of the acting forces has no component in a direction perpendicular to the straignt line on which the body moves.

Example (2) :

Three bodies of masses 1, 2, 3 kg are let to fall.

Find the magnitude of the force acting on each, neglecting air resistance to its motion.

Solution :

Neglecting air resistance, the only force acting on each body is the force of its weight acting vertically downwards as in fig. (19). The magnitude of this force can be obtained from Newton's second law. The first body :

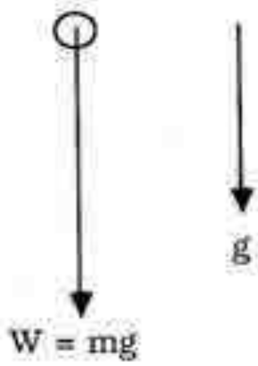


Fig. (19)

$F_1 = m_1 g = 1 \times 9.8$

$= 9.8$ newtons $= 1$ kg. wt.

The second body

$F_2 = m_2 g = 2 \times 9.8$ newtons $= 2$ kg. wt.

The third body

$F_3 = m_3 g = 3 \times 9.8$ newtons $= 3$ kg. wt.

We could have obtained these results directly by noticing that the magnitude of the weight (measured in kg. wt) is numerically equal to the magnitude of the mass (measured in units of kg.).

Example (3) :

A car of mass 1.8 tons moves along a straight horizontal road with a velocity of 60 km./h The car stopped its motor, and it continued its motion a distance of 200 metres until it stopped completely. Find the magnitude of the resisting force assuming that it is constant during the motion of the car.

Solution :

Let A be the position where the car's motor stopped, B be the position where the car stopped. It is clear that the motion from A to B is under the effect of a single force which is the force of resistance, since the resultant of the two other forces, the weigh of the car and the normal reaction vanish.

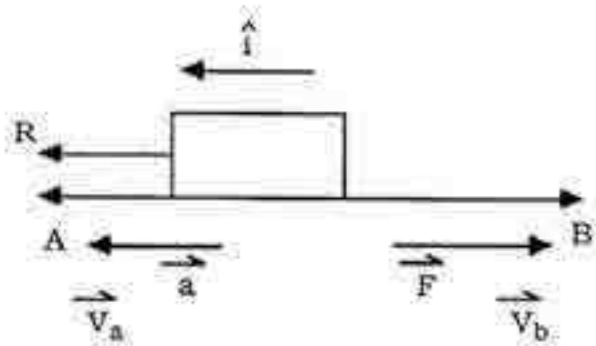


Fig. (20)

Let $\hat{i}$ be a unit vector in a direction opposite to that of motion, let a , s be the algebraic measures of the acceleration and displacement vectors respectively. Fig. (20). It is clear that

$$s = -200 \text{ metres.}$$

Applying the well known law

$$V^2_b - V^2_a = 2as$$

Taking $V_a = -60 \text{ km/h} = -60 \times \frac{10^3}{3600} = \frac{-10^2}{6}$ m/sec.

$V_b = 0$

$$\therefore 0 - \left(\frac{-10^2}{6}\right)^2 = 2a(-200)$$

$$\therefore a \frac{25}{36} = \text{m/sec}^2$$

The magnitude o the resisting force is obtained from Newton's second law.

$R = m\,a$ "m is measured in kilogram"

$= 1.8 \times 10^3 \times \frac{25}{36}$

$= 1250$ newtons.

Example (4) :

A force of magnitude F acts on a body placed on a smooth horizontal table in a direction making an angle of mesure θ with the downward vertical. Find the acceleration of the body as a result of this action. Find also the magnitude of the normal reaction of the table.

Solution :

Since the table is smooth, the resistance force vanishes. Let N be the normal reaction, a is the magnitude of the produced acceleration. Fig (21). Since the motion is horizontal the sum of the components or the forces in the vertical direction vanishes.

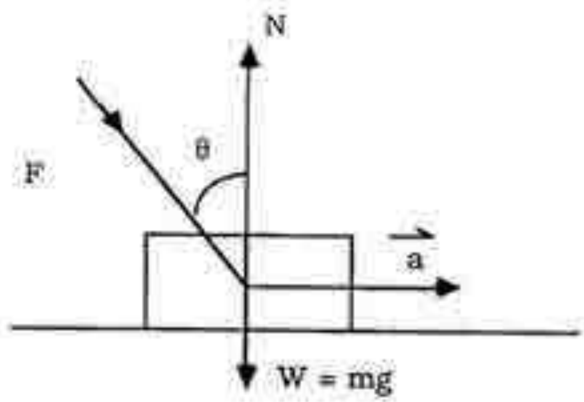


Fig. (21)

$\therefore \quad N - m\,g - F\cos\theta = 0$

$\therefore \quad N = m\,g + F\cos\theta$

This relation determines N, and it is clear that the magnitude of the reaction differs from the weight, and is equal to it only when $\cos\theta = 0$ i.e when the force F is horizontal.

From Newton's second law in its scalar form, we find

$$m\,a = F\sin\theta$$

$$\therefore \quad a = \frac{F}{m}\sin\theta$$

Therefore the body moves with a constant acceleration. It is clear that the magnitude of a increases when θ increases, and becomes maximum when $\sin\theta = 1$ i.e. when F is horizontal.

Example (5) :

A flying body of mass 800 kg flies in space with uniform motion with speed of 900 km/h. Suddenly it enters a cloud of dust which acts on it with a force of friction (resistance) whose magnitude is 1/2 kg. wt for each kilogram of its mass. Find its velocity after coming out of the cloud, if it stayed inside the cloud for 20 seconds.

Solution :

When the body is inside the cloud, a resistance force acts on it in a direction opposite to its direction of motion. Let R be the magnitude of this force Fig. (22) $R = 1/2 \times 800 = 400$ kg. wt. $= 400 \times 9.8$ newtons.

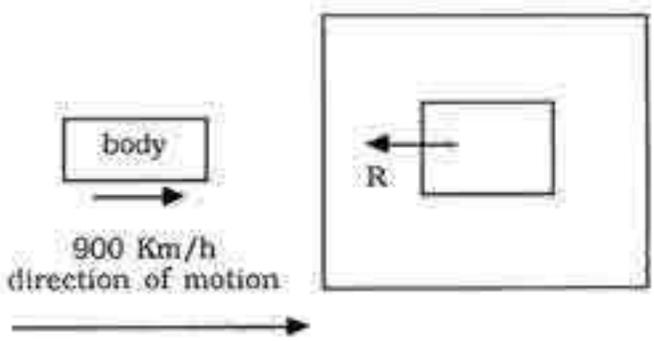


Fig. (22)

This force cause a retardation motion with an acceleration a which can be calculated from Newton's second law.

$$F \text{ (newton)} = m \text{ (kg)} \times a \text{ (m/sec}^2\text{)}$$

$$\therefore 400 \times 9.8 = 800\, a$$

$$\therefore a = \frac{400 \times 9.8}{800} = 4.9 \text{ m/sec}^2$$

$$\therefore a = \frac{4.9 \times 10^{-3}}{\frac{1}{3600} \times \frac{1}{3600}} = 49 \times 36 \times 36 \text{ km/h}^2$$

The body moves with this acceleration for 20 seconds

i.e. $\frac{20}{3600}$ hour.

Applying the well known law

$$V = V_o - a\,t$$

putting $V_o = 900$ we get

$$V = 900 - (49 \times 36 \times 36) \times \left(\frac{20}{3600}\right)$$

$$= 547.2 \text{ km/h.}$$

Example (6) :

A body of variable mass, whose mass is $m = a\,t + 1$, moves along a fixed straight line. If its displacement vector is given by the relation $\bar{s} = (1/2\ t^2 + t)\ \hat{i}$, where $\hat{i}$ is a unit vector parallel to the straight line. Find the momentum of this body, and deduce the law of the force acting on it.

Solution :

$$m = 2t + 1$$

velocity vector $\vec{v} = \frac{d\vec{s}}{dt} = \frac{d}{dt}(1/2\ t^2 + t)\ \hat{i}$

$$= (t + 1)\ \hat{i}$$

Momentum vector $\vec{H} = m\,\vec{v} = (2t + 1)(t + 1)\ \hat{i}$

$$= (2\,t^2 + 3\,t + 1)\ \hat{i}$$

From the second law

$$\vec{F} = \frac{d}{dt}\ (m\ \vec{v}) = \frac{d\vec{H}}{dt} = \frac{d}{dt}\ (2\ t^2 + 3\ t + 1))\ \hat{i}$$

$$= (4\ t + 3)\ \hat{i}$$

i.e. the acting force on the body is of magnitude $(4\ t + 3)$ and in direction of the unit vector $\hat{i}$.

Exercises (1 - 3)

1) A particle of mass m moves under the action of two forces, $\overrightarrow{F_1} = 3\,m\,\hat{i}$, $\overrightarrow{F_2} = 5\,m\,\hat{j} - 2\,m\,\hat{i}$ where $\hat{i}$, $\hat{j}$ are two perpendicular unit vectors. Find the acceleration vector and find its magnitude.

2) A particle of unit mass is moving so that its velocity vector is given as a function of th time t in the form $\vec{v} = (A\,t^2 + B\,t)\,\hat{i}$, where $\hat{j}$ is a constant unit vector. Find the constants A, B if the force acting on this partile is constant and is given by the relation $\vec{F} = 5\,\hat{i}$.

3) A particle of unit mass is moving under the effect of three forces. $\overrightarrow{F_1} = \hat{i} + A\,\hat{j}$, $\overrightarrow{F_2} = -2\,\hat{i} + \hat{j}$, $\overrightarrow{F_3} = 2\,\hat{j} + B\,\hat{i}$, where $\hat{i}$, $\hat{j}$ are two perpendicular unit vectors, A, B are constants. If the displacement vector of the particle is given as a function of the time t in the form, $\vec{s} = \hat{i} + (1/2\,t^2 + t)\,\hat{j}$, find the constants A, B.

4) A particle of mass 1 kg moves such that the components of its velocity in the horizontal and vertical upwards directions are respectively $v_x = 2$, $v_y = -9.8\,t + 2$ in units of m/sec. Find the magnitude and direction of the intitial velocity of this particle, and the force vector acting on it.

5) A force of magnitude 100 newtons acts on a body of mass 20 kg. in a direction making an angle of measure 30° with the vertical downwards. If the body is placed on a smooth horizontal ground. find the acceleration produced and the magnitde of the normal reaction.

6) A tank of mass 20 tons and the force of its machine is 1/2 ton weight, starts moving on a horizontal ground. If the resistance force to its motion is equal in magnitude to 20 kg. wt per ton of its mass. find the tank's velocity after 250 seconds from the start of the motion.

7) A body in the form of a right circular cylinder, of height 50 cm. and bases radius 10 cm. and mass 10 kg is moving uniformly with velocity 5 m/sec. This body enters a dusty cloud, which acts on it with a resistance force of magnitude 0.01 gm. wt to every square centimeter of its lateral surface. Find the velocity of the body after getting out of the cloud, given that it moved inside it for 30 seconds.

8) A bullet of mass 25 gm and moving with velocity 200 m/sec. is fired at a fixed barrier, it moved inside it for 5 cm, until it stopped. Find the resistance force of the barrier to the motion of the bullet, given that this re[illegible]ce remained constant all the time.

9) A metal ball of mass 100 gm moves along a straight line with a uniform velocity 10 m/sec. in a dusty medium. If dust adher as to its surface at the rate of 0.06 gm per second, find the mass of the ball and the force acting on it at any time t, given that at the beginning of motion the ball was completely free from any dust.

10) A car of mass 1960 kg moves with velocity 63 km/h. If the resistance to the motion of the car is 1250 Kg. wt. Find the distance described by the car before it stops.

11) A constant horizontal force of 1000 kg. weight acts on a car of mass 4 tons moving on a horizontal road. If the car starts from rest and its velocity amounted to 4.9 m/sec. after 10 seconds, find the magnitude of the friction force.

12) A mass of 2 kg falls 10 m from rest and is then brought to rest by penetrating 5 cm into some sand, find in kg. wt the thrust of the sand supposing it to be uniform.

13) A force $\vec{F}$ acts upon a body of mass 3 kg. The body covered a distance of 245 cm in 10 seconds against a resistance force to its motion equals 1/10 of its weight.Find $\vec{F}$. If $\vec{F} = \vec{0}$ at the end of this interval and the resistance did not change. Find the time taken by the body till it comes to rest.

14) A train of mass 245 tons (the mass of th engine and the train)moves with unifrom acceleration 15 cm/sec^2. If the

resistance of air and friction is 4 kg. wt per ton, find the force of the engine. If the last car of the train of mass 49 tons is released after the train moves from rest for 4.9 minutes, find the acceleration of the train and time taken by the relased car till it comes to rest.

15) A balloon of mass 1050 kg ascending vertically upwards with uniform velocity. A body of mass 70 kg falls from it. Find the acceleration with which the balloon moves. If the velocity of the balloon before the body falled is 50 cm/sec. find :

$1^{\underline{st}}$: the distance covered by the balloon in the next 10 seconds.

$2^{\underline{nd}}$: the distance between the balloon and the body after this interval.

16) A body of unit mass is moving under the effect of a force $\overrightarrow{F} = a\,\hat{i} + b\,\hat{j}$, where $\hat{i}$ and $\hat{j}$ are two orthogonal unit vectors. If the displacement vector of the body is given by the relation :

$$\overrightarrow{S} = (2t^2 + 1)\,\hat{i} + (3t^2 + 1)\,\hat{j},$$

where t is time. Find a and b.

Newton's Third Law :

To every action there is a reaction equal in magnitude and opposite in direction.

If body (1) acts on body (2) With a force $\overrightarrow{F_{21}}$, then body (2) Acts on body (1) with a force $\overrightarrow{F_{12}}$, these two forces satisfy The relation (fig. 23)

$\overrightarrow{F_{12}} = -\overrightarrow{F_{21}}$

i.e. $\overrightarrow{F_{12}} + \overrightarrow{F_{21}} = \overrightarrow{O}$

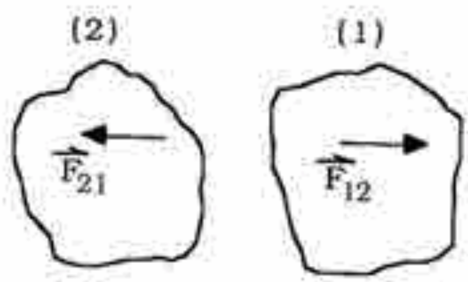


Fig. (23)

This relation enables us to express Newton's third law in a new form

"The resultant of mutual forces between any two bodies vanishes".

The last relation indicates that the two forces $\overrightarrow{F_{12}}$, $\overrightarrow{F_{21}}$ are parallel, and thus we can express each in terms of their algebraic measures.

$$\text{If} \quad \overrightarrow{F_{12}} = F_{12}\ \hat{c} \qquad , \qquad \overrightarrow{F_{21}} = F_{21}\ \hat{c}$$

where $\hat{c}$ is a unit vector parallel to the two forces, Newton's third law takes the simple form :

$$\overrightarrow{F_{12}} + \overrightarrow{F_{21}} = \overrightarrow{0}$$

i.e. $\quad F_{12} + F_{21} = 0$

which means that "The sum of the algebraic measures of the mutual forces between any two bodies vanishes".

Discussion of the third law :

1) Newton's first and second laws explain how forces affect bodies, while the third law determines to us the rule which the mutual forces between two bodies follow.

2) Experiment supports the truth of the third law, for example if you press a table with your finger you will feel the table's

pressure on you finger, and when you increase your pressure on the table, the pressue will increase on your finger.

3) The vanishing of the resultant ($\overrightarrow{F_{12}} + \overrightarrow{F_{21}}$) does not mean that the effect of one of the two forces cancels the effect of the other force, for each one of them acts on one body only, and not the other, and if we refer to fig. (23) we find that the force $\overrightarrow{F_{12}}$ acts on body (1), the force $\overrightarrow{F_{21}}$ acts on body (2).

Illustrative example :

A body at rest on a horizontal table.

Consider a body at rest on a horizontal table. Due to the contact between the body and the table, both act on the other with a force, and the two forces satisfy Newton's third law.

We knew before that the table acts on the body with a force called the resultant raction force, and of magnitude N.

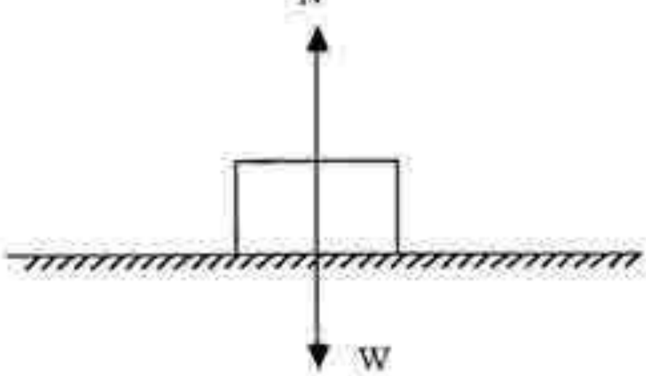


Fig. (24)

Since the body is in equilibrium under the action of this force and the force of its weight, which is directed vertically downwards. Therefore the resultant reaction force is directed vertically upwards. i.e. it is equal to the normal reaction (fig. 24). and if W is the weight of the body, then from equilibrium we have

$$N = W$$

But if we consider the effect of the body on the table, it is a force satisfying Newton's third law, i.e. it is force directed vertically downwards, and of magnitude equal to the weight of the body (fig. 25).

This force is called the pressure of the body on th table, and is denoted by P. i.e. $P = W$

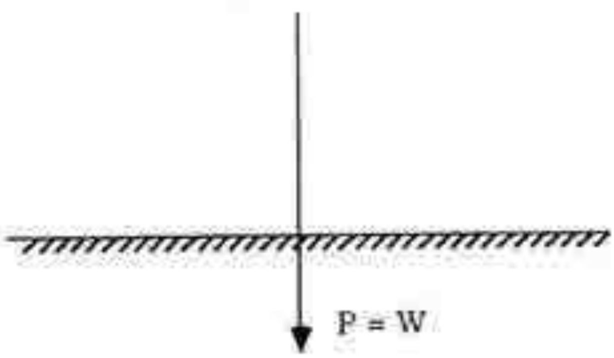


Fig. (25)

Simple Applications on Newton's Laws of Motion

I. A body placed inside a lift moving with a uniform acceleration:

a) The body placed on the floor of the lift :

Consider a body of mass m placed on the floor of a lift moving vertically with a uniform acceleration of magnitude a.

Since the body is at rest inside the lift, then it will acquire the same acceleration.

The forces acting on the body are :

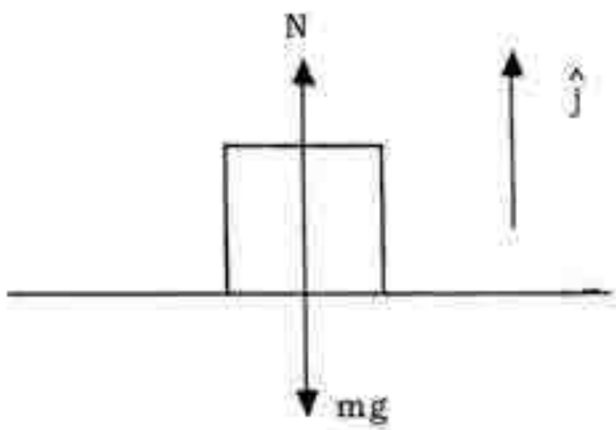


Fig. (26)

1. The weight force (denoted by $\overrightarrow{F_1}$) of magnitude mg acting vertically downwards.
2. The reaction force, which is the effect of the bottom of the lift on the body, (denoted by $\overrightarrow{F_2}$) of magnitude N acting vertically upwards, since there is no force of friction between the body and the bottom of the lift, the body being at rest on the bottom.

   Let $\hat{j}$ be a unit vector directed vertically upwards (Fig. 26).

We can write each of the two forces in terms of $\hat{j}$ as follows :

$$\overrightarrow{F_1} = - m g \hat{j} \qquad \overrightarrow{F_2} = N \hat{j}$$

The acceleration of the body is that of the lift, and can be written in the form

$$\vec{a} = \pm a \hat{j}$$

The positive sign is taken when the lift is moving upwards, and the negative sign is taken if it is moving downwards.

Applying Newton's second law on the motion of the body

$$m \vec{a} = \overrightarrow{F_1} + \overrightarrow{F_2} \qquad (1)$$

Consider the following three cases :

1. The lift is at rest or moving with a uniform velocity. In this case we put $\vec{a} = \vec{0}$ in equation (1)

$\therefore \quad \overrightarrow{F_1} + \overrightarrow{F_2} = \vec{0}$

$\therefore \quad - m g \hat{j} + N \hat{j} = \vec{0}$

$\therefore \quad (- m g + N) \hat{j} = \vec{0}$

$\therefore \quad - m g + N = 0$

$\therefore \quad N = m g \qquad (2)$

which is a well known relation to us, and shows that the reaction force of the bottom of the lift on the body (or the magnitude of the pressure force of the body on the bottom of the lift) is equal to the weight of the body.

2. The lift moving with acceleration in the direction upwards :

In this case $\vec{a} = a\ \hat{j}$.

Substituting this value in equation (1), we get

$$m\,a\,\hat{j} = \overrightarrow{F_1} + \overrightarrow{F_2} = -\,m\,g\,\hat{j} + N\,\hat{j}$$

Eliminating $\hat{j}$ from both sides

$$\therefore \quad m\,a = -\,m\,g + N$$

$$\therefore \quad N = m\,(g + a) \qquad (3)$$

By comparing (3), (2) we notice that the magnitude of the reaction force of the bottom on the body (or the magnitude of the pressure force of the body on the bottom of the lift) has increased by the value (m a) than its value when the lift was at rest or moving with a uniform velocity, and we express that by saying that "the body inside a lift moving upwards with a uniform acceleration of magnitude a feels as if the gravitational acceleration has increased by the amount a"

3. The lift moving with acceleration in the direction downwards :

In this case $\vec{a} = -\,a\ \hat{j}$.

Substituting this value in (1), we get

$$-\,m\,a\,\hat{j} = \overrightarrow{F_1} + \overrightarrow{F_2} = -\,m\,g\,\hat{j} = N\,\hat{j}$$

$$\therefore \quad -\,m\,a = -\,m\,g + N$$

$$\therefore \quad N = m\,(g - a) \qquad (4)$$

By comparing (4) and (2) we notice that the magnitude of the reaction force of the bottom on the body (or the magnitude of the pressure force of the bbdy on the bottom on the lift) has decreased by teh value (m a) than its value when the lift was at rest or moving with a uniform velocity, and we express that by saying that "the body inside the lift feels as if the gravitational acceleration has decreased by the amount a when the lift is moving downwards with uniform acceleration".

When the magnitude of the acceleration with which the lift descends increases, the magnitude of the reaction force decreases, until when the lift descends with the gravitational acceleration (case of free descent), the magnitude of reaction force vanishes, since each of the body and lift are descending with the same acceleration, which is the gravitational acceleration, and the body is about to leave the bottom.

b) Spring balance suspended from the top of a lift :

Consider now a body of mass m attached to the end of a spring balance suspended from the top of a lift moving vertically with a uniform acceleration of magnitude a.

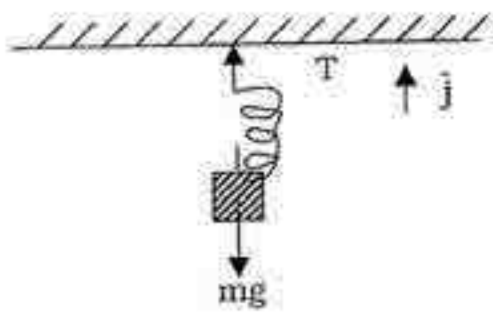


Fig. (27)

Since the body and the balance are fixed inside the lift, then they acquire the same acceleration. The forces acting on the body are :

1. The weight force (denoted by $\overrightarrow{F_1}$), of magnitude m g and acting vertically downwards.
2. The tension force of the spring to the body (denoted by $\overrightarrow{F_2}$), directed vertically upwards, let its magnitude be T, which is the same in magnitude as the pull of the body of the spring directed downwards.

Let $\hat{j}$ be a unit vector directed vertically upwards.

$\therefore \overrightarrow{F_1} = - m g \hat{j} \quad , \quad \overrightarrow{F_2} = T \hat{j}$

If $\vec{a}$ is the acceleration vector of the body (or the lift, then applying the second law on the body,

$m \vec{a} = \overrightarrow{F_1} + \overrightarrow{F_2}$

We can now apply the results of the last item putting the tension force in place of the reaction force.

Consider the following three cases :

i. The lift at rest or moving with a uniform velocity

$T = m g \qquad (5)$

i.e. the magnitude of the tension force is equal to the weight of the body.

ii. Lift moving with acceleration downwards :

$T = m\ (g + a)$ (6)

i.e. the magnitude of the tension force increases by (m a) than in case (1)

iii. Lift moving with acceleration upwards :

$T = m\ (g - a)$ (7)

i.e. the magnitude of the tension force decreases by (m a) than in case (1).

Example (1) :

Find the reaction of a lift on a person inside it, whose mass is 70kg in newtons in the following cases :

1. If the lift is moving with a uniform velocity.
2. If the lift is moving vertically upwards with a uniform acceleration of magnitude 1.2 m/sec^2.
3. If the lift is moving vertically downwards with a uniform acceleration of magnitude 1.8 $m./sec^2$.

Solution :

1. $\therefore$ N = mg.

$\therefore$ N = 70 × 9.8 = 686 newtons

2. $\therefore$ N = m (g + a)

$\therefore$ N = 70 (9.8 + 1.2) = 70 × 11.

= 770 newtons.

3. $\therefore$ N = m (g - a)

$\therefore$ N = m (g - a)

$\therefore$ N = 70 (9.8 - 1.8) = 70 × 8 = 560 newtons

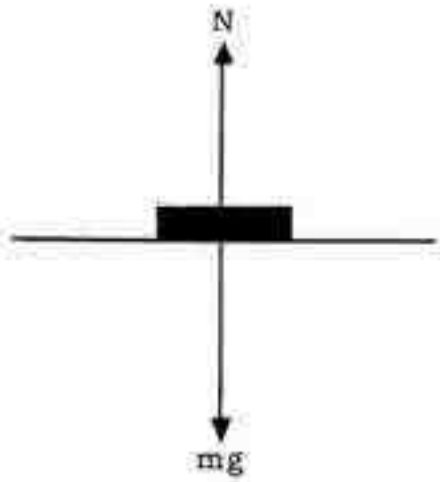


Fig. (28)

Example (2) :

A spring balance carrying a body of mass 35 kg is suspended from the top of a lift moving vertically with a uniform acceleration of magnitude 140cm./sec^2

Find the apparent weight indicated by the balance in Kg.wt

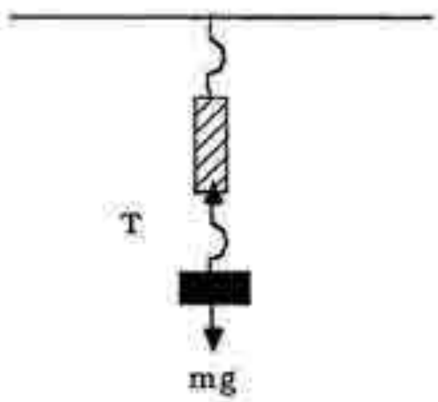


Fig. (29)

i. if the direction of the acceleration of the lift is upwards.

ii. if the direction of the acceleration of the lift is downwards.

Solution :

The apparent weight is the reading of the spring balance (i.e. the tension).

1. $\therefore \quad T = m\,(g + a)$

$\therefore \quad T = 35 \times 1000 \times (980 + 140)$

$= 35 \times 1000 \times 1120$ dynes

$= \dfrac{35 \times 1000 \times 1120}{980 \times 1000} = 40$ kg.wt.

$\therefore \quad T = m\,(g - a)$

$\therefore \quad T = 35 \times 1000 \times (980 - 140)$

$= 35 \times 1000 \times 840$ dynes

$= \dfrac{35 \times 1000 \times 840}{980 \times 1000} = 30$ kg.wt.

II. Motion of a Body on a Smooth Inclined Plane:

Consider a body of mass m moving along a line of greatest slope of a smooth inclined plane whose inclination to the horizontal is α under the action of a force of magnitude F in the direction of a line of greatest slope.

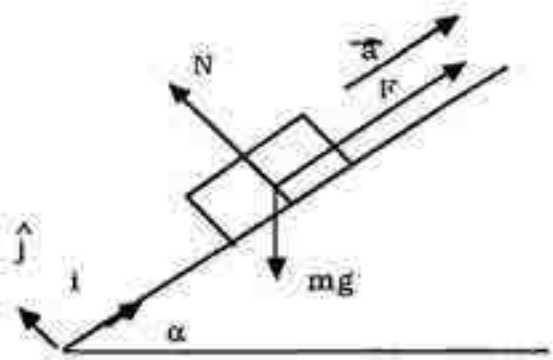


Fig. (30)

The forces acting on the body (Fig. 30) are :

1. The given force acting along a line of greatest slope, and of magnitude F.
2. The force of weight acting vertically downwards, and of magnitude m g (g is the gravitational acceleration). This force can be resolved into two components, one in the direction of a line of greatest slope downwards and of magnitude (m g sin α), and the other in a direction normal to the plane and towards it of magnitude (m g cos α) (Fig. 31).

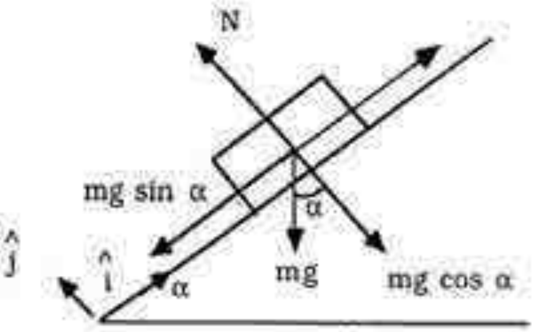


Fig. (31)

3. The force with which the plane acts on the body (reaction force); acting in a direction normal to the plane and upwards (notice that the plane is smooth, therefore there is no force of friction). This force was called before "normal reaction", let N be its magnitude, and this is an unknown to be determined. It is useful to assume two perpendicular unit vectors $\hat{i}$, $\hat{j}$, then first along a line of greatest slope upwards, and the second normal to the plane upwards (Fig. 30).

Let $\vec{a}$ be the acceleration vector of the body. It is clear that this vector is parallel to the vector $\hat{i}$, since motion is parallel to this vector.

$\therefore \quad \vec{a} = a\,\hat{i} \qquad (1)$

When a here is the algebraic measure of the vector $\bar{a}$ relative to the unit vector $\hat{i}$, $a > 0$ in the case when the motion on the plane is upwards.

i.e. in direction of $\hat{i}$, while when $a < 0$ in the case when the motion on the plane is downwards, i.e., in an opposite direction to $\hat{i}$. If the motion is uniform then $\vec{a} = \vec{0}$.

Since the motion is parallel to the vector $\hat{i}$, the sum of the components of the forces in direction of $\hat{j}$ must vanish.

$\therefore \quad N - m\,g\cos\alpha = 0$

From this equation we can determine the magnitude of the normal reaction.

$N = m\,g\cos\alpha \qquad (2)$

Applying Newton's second law to the motion of the body parallel to the vector $\hat{i}$, we find

$m\,a = F - m\,g\sin\alpha$

$\therefore \quad a = \frac{F}{m} - g\sin\alpha \qquad (3)$

In the special case in which the force $\vec{F}$ vanishes $(F = 0)$ then

$a = -\,g\sin\alpha$

Example (3) :

A body of mass 1 kg is placed on a smooth inclined plane of inclination 60° to the horizontal. A horizontal force of magnitude 10 kg.wt towards the plane, and its line of action lies in the vertical plane through the line of greatest slope to the plane acts on the body. Find the acceleraiton produced and the magnitude of the normal reaction.

Solution :

The forces acting on the body are shown in Fig. (32). Consider two unit vectors $\hat{i}$, $\hat{j}$ as shown in figure.

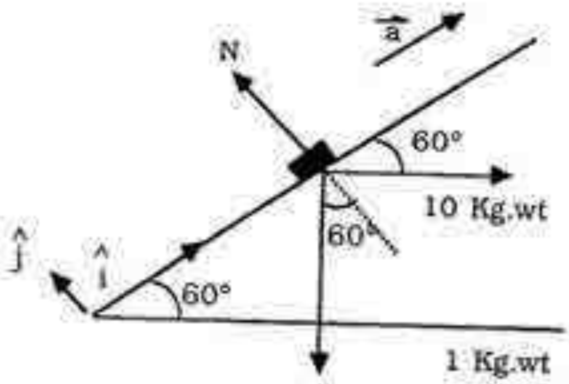


Fig. (32)

Resolving the acting forces in the direction of the plane and the normal to it, we get figure (33).

Since the sum of the components of the forces in direction of $\hat{j}$ is equal to zero.

$\therefore \quad N - (10 \sin 60 + 1 \cos 60) = 0$

$\therefore \quad N - 5\sqrt{3} - 1/2 = 0$

$\therefore \quad N = 5\sqrt{3} + 1/2 = 9.16$ kg.wt.

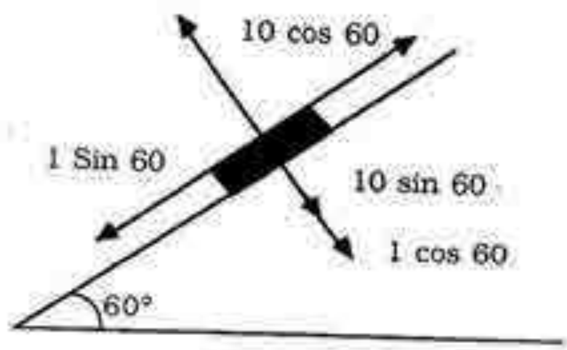


Fig. (33)

Applying Newton's second law to the motion in direction of $\hat{i}$, we get $1 \times a = (10 \cos 60 - 1 \sin 60) \times 9.8$ (notice that the magnitudes of the forces are in units of newton and the magnitude of the acceleration in units of m/sec^2).

$$\therefore \quad a = (5 - \frac{\sqrt{3}}{2}) \times 9.8$$

$$\simeq (5 - 0.866) \times 9.8$$

$$\simeq 4.134 \times 9.8 \simeq 40.5 \quad m/sec^2$$

i.e. the acceleration vector of the body is in direction of $\hat{i}$, i.e. upwards along a line of greatest slope.

Exercises (1 - 4)

1. Find the reaction of a lift on a person inside it, whose mass is 60kg in units of newton in the following cases :
   a) if the lift is at rest.
   b) if the lift moves vertically upwards with a uniform acceleration of magnitude 1.7 m/sec^2.
   c) if the lift moves vertically downwards with a uniform acceleration of magnitude 2.8 m/sec^2.
2. A body of mass 2 kg is placed on the floor of a lift. Find the pressure force of this body on the floor of the lift, when the left;
   a) is moving with a uniform velocity.
   b) is moving upwards with an acceleration of magnitude 98cm/sec^2.
   c) is moving downwards with an acceleration of magnitude 98$cm./sec^2$.
3. An electric lift is ascending with an acceleration of magnitude 70cm/sec^2. A man inside this lift is pressing with his leg on the floor of the lift with a force of magnitude 67.5 kg.wt. Find the mass of this man.
4. A body of mass 1 kg is suspended from the end of a spring balance fixed in the ceiling of a lift. The lift moves with a uniform acceleration, the balance reading was 800 gm.wt. Find the magnitude of the acceleration of the lift and its direction.

5. A spring balance is fixed in a lift moving vertically. A body of mass 490gm is suspended from the end of the balance. If the balance reading is 450 gm.wt. is the lift ascending or descending? What is the magnitude of its acceleration ?

6. A body is suspended from the end of a spring balance, fixed in the ceiling of a lift. When the lift was moving upwards with an acceleration of magnitude a cm/sec^2, the balance reading was 16 kg.wt. and when it was moving upwards with an acceleration of magnitude a cm/sec^2, the balance reading was 17 kg.wt. Find the mass of the body and the magnitude of a. Find also the balance reading when the lift is descending with a uniform retardation of magnitude $\frac{3}{2}$ a.

7. A body is suspended from the end of a spiral fixed in the ceilling of a lift. The readings of the balance when the lift was moving upwards with a certain acceleration then when moving downwards with the same acceleration were 1.22 kg.wt 0.78 kg.wt respectively.

Find the mass of the body and the magnitude of the lift's acceleration.

8. A body of mass $1/2$ kg is placed on a smooth plane inclined to the horizontal at an angle of $30°$, then is left to move. Find the magnitude of the force of reaction of the plane on the body. Find also the magnitude of its acceleration on the plane.

9. A body of mass 1 kg is placed on a smooth plane inclined to the horizontal at an angle of measure $30°$. A force of magnitude 10 newtons acts on the body along a line of greatest slope upwards. Find the magnitude of the force of reaction of the plane on it, and find also its acceleration.

10. A body of mass 2 kg is moving along a line of greatest slope of a smooth plane inclined at an angle of measure $60°$ to the horizontal under the action of a force of magnitude 1 kg.wt directed towards the plane and making an angle of measure $30°$ with the horizontal upwards. Find the magnitude of the reaction force on the body, and also its acceleration.

11. A body is projected up on a smooth inclined plane at an angle of measure α, where $\sin \alpha = 0.1$, the velocity of projection is 49 cm/sec. Find the time taken to return the body to the point of projection.

12. A body of mass 500 gm is placed on a smooth plane inclined to the horizontal at an angle of measure θ, $\sin\theta = 3/5$ Forces of 500 gm.wt acting parallel to the plane acts on the body. Find the acceleration of motion if the force equal zero after 2 seconds. Find the distance which the body ascends untill it stops instantaneously.

13. A car starts its motion from rest descending an inclined plane to the horizontal at an angle of measure θ. $\sin\theta = 1/100$, its velocity amounted to 44.1 km per hour after 250 sec. Calculate the resistance per ton.

14. A train of mass 240 tons moves in a horizontal road with a uniform acceleration 2.45 cm/sec^2 If the force of the engine is 2000 kg.wt, find the resistance per ton of the mass of the train and if the train ascends an inclined plane of 1 in 500, find the acceleration of the train given that the resistance does not change.

15. A rough inclined plane its length is 40m, and its height is 10m. If a body is projected up the plane from the bottom, find the least velocity by which it must be projected to reach the top of the plane given that the force of the friction of the plane is 1/4 of the weight of the body.

Chapter Two

Applications on Newton's laws
Motion on a rough plane

Preface:

In this chapter we will deal with: Some Applications on Newton's laws of motion.

Objectives:

By the end of teaching this chapter, the student should be able to:

(1) Find the acceleration of motion of two bodies connected by the two ends of a string passing over a smooth pulley , the tension in the string and the pressure on the axis of the pulley.

(2) Find the acceleration of motion of two bodies connected by the two ends of a string passing over a smooth pulley at the edge of a smooth table .

(3) Find the acceleration of motion of two bodies connected by the two ends of a string passing over a smooth pulley at the edge of the top of a smooth inclined plane .

(4) Find the acceleration of motion of two bodies connected by the two ends of a string passing over a smooth pulley at the edge of a rough horizontal table .

(5) Find the acceleration of motion of two bodies connected by the two ends of a string passing over a smooth pulley at the edge of the top of a rough inclined plane .

Chapter Two

Applications on Newton's laws
Motion on a rough plane

Topics :

(1) Motion of a set of two bodies suspended vertically from the two ends of a string passing over a smooth pulley .

(2) Motion of a set of two bodies connected by a string one of them moving on a smooth horizontal plane and the other moving vertically .

(3) Motion of a set of two bodies connected to the two ends of a string ,one of them moving on a smooth inclined plane and the other moving vertically .

(4) Motion of a set of two bodies connected to the two ends of a string ,one of them moving on a rough horizontal plane and the other moving vertically .

(5) Motion of a set of two bodies connected to the two ends of a string ,one of them moving on a rough inclined plane and the other moving vertically .

In this chapter we give some applications of Newton's laws of motion, which relate to the motion of two bodies connected together by strings. We assume that the length of a string remains constant. We also assume that the weight of a string is too small compared to the weight of the connected bodies, so that we may neglect its weight.

First application :

Motion of a system of two bodies hanging vertically from the ends of a string which passes over a pulley.

Consider two bodies, of masses m_1 and m_2 where $m_1 > m_2$ connected together by a string of constant length (i.e. inextensible) and of negligible weight. The string passes over a small smooth pulley, we may thus neglect the length of that part of the string in contact with the pulley, in comparison with the complete length of the string.

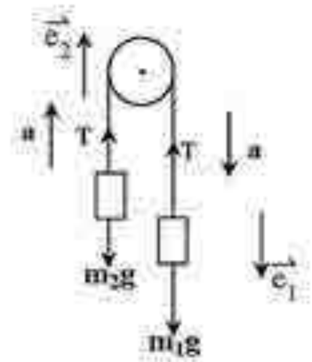


Figure (34)

We assume that both parts of the string hang vertically as shown in Fig.(34).The system is allowed to move and we are interested in the resulting motion.

To study this problem, we consider the motion of each body separately. Since m_1 is larger than m_2 we expect m_1 to move vertically downwards, and

m_2 to move vertically upwards. Let $\overrightarrow{e_1}, \overrightarrow{e_2}$ be unit vectors directed respectively along the downward and upward verticals. Let a be the algebraic measure relative to $\overrightarrow{e_1}$ of the acceleration vector of body of mass m_1. Since the string is inextensible, we deduce that the algebraic measure relative to $\overrightarrow{e_2}$ of the acceleration vector of body of mass m_2 is also a.

Equation of motion of the body of mass m_1:

$$m_1 a = m_1 g - T \qquad (1)$$

Equation of motion of the body of mass m_2:

$$m_2 a = T - m_2 g \qquad (2)$$

Where T denotes the magnitude of the tension in either parts of the string, g denotes the magnitude of the acceleration of gravity.

Adding equations (1) and (2), we get

$$(m_1 + m_2)\, a = (m_1 - m_2) g$$

From which we determine a:

$$a = \frac{m_1 - m_2}{m_1 + m_2} \times g \qquad (3)$$

We see from this relation that the body of larger mass moves vertically downwards.

Remark :

Had we assumed from the state that the body of m_2 has acceleration directed vertically upwards, we would have ended with a negative value for a.

If the two bodies were equal, the system will remain static or each of the bodies will have a uniform motion with the same magnitude of velocity.

We may obtain the tension in the string from equation (1) say, after substituting for a its obtained value.

If $\hat{e}$ is a unit vector, B is parallel to $\hat{e}$, we write $\overrightarrow{B} = B\,\hat{e}$ where is called the algebraic measure of $\overrightarrow{B}$ relative to $\hat{e}$. Notice that if $\overrightarrow{B}$, $\hat{e}$ have the same direction $\overrightarrow{B}$ positive, otherwise it is negative.

The pressure on the pulley :

The string exerts two forces on the pulley each of magnitude T, which are directed vertically downwards, as in Fig.(35). The resultant of these two forces is called the pressure on the pulley, its magnitude is:

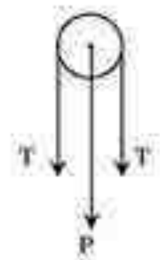


Figure (35)

$$P = 2\,T$$

Remark :

If the direction of motion for both bodies is known, we may neglect introducing the unit vectors $\hat{e_1}$, $\hat{e_2}$ as will be done in the following examples.

Example (1) :

Two bodies of masses of 1 , 3kg are connected to the ends of a string which passes around a small smooth pulley. Find the acceleration of the system and the pressure on the pulley.

Solution :

Suppose that the body of mass 3kg is accelerated vertically downwards with a magnitude "a", consequently the body of mass 1kg accelerates vertically upwards, with the same magnitude "a", as shown in Fig. (36). Since the acceleration of gravity $g = 9.8\ m/sec^2$

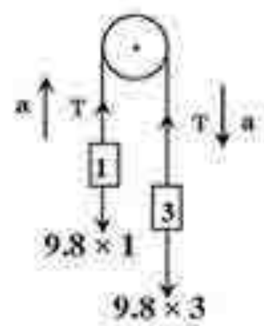


Figure (36)

the equation of motion for the body of mass 3kg is

$$3a = 3 \times 9.8 - T \qquad (1)$$

the equation of motion for the body of mass 1kg is

$$a = T - 9.8 \qquad (2)$$

Adding (1) and (2)

$$4a = 19.6$$

$$a = \frac{19.6}{4} = 4.9 \text{ m/sec}^2$$

To obtain the tension in the string, we substitute for "a" in equation (1)

$$T = 3 \times 9.8 - 3 \times 4.9$$

$$= 14.7 \text{ newtons}$$

The pressure on the pulley is twice the tension

$$P = 2T = 29.4 \text{ newtons}$$

Example (2) :

Two bodies of masses of 3 , 5kg are tied at the two ends of a string which passes round a small smooth pulley. The system is kept in equilibrium with the two parts of the string hanging vertically. If the system was left to move, find the magnitude of its acceleration and the pressure on pulley.

Find also the speed of the body of larger mass when it has descended 40cm.

Solution :

We know that the body of larger of the masses will move vertically downwards as shown in Fig.(37). The magnitude of the acceleration of gravity is $g = 9.8$ m/sec$_2$.

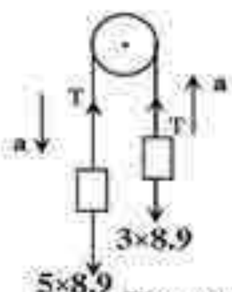


Figure (37)

the equation of motion of the large mass is :

$$5a = 5 \times 9.8 - T \qquad (1)$$

the equation of motion of the large mass is :

$$3a = T - 3 \times 9.8 \qquad (2)$$

Adding (1) and (2)

$$8a = 2 \times 9.8$$

$$a = 2.45 \text{ m/sec}^2$$

We determine the tension in the string from equation (1)

$$T = 5 \times 9.8 - 2.45 \times 5 = 36.75 \text{ newtons}$$

The pressure on the pulley is :

$$P = 2T = 73.5 \text{ newtons}$$

If V is the speed of the body of large mass after descending 40cm, we have :

$$V^2 = 2 \times 2.45 \times 0.4 = 1.96$$

$$V = 1.4 \text{ m/sec.}$$

Second Application :

The motion of a system of two bodies, one of them moving on a smooth horizontal plane, the other moving vertically.

Consider two bodies m_1 and m_2 connected by a string, the body of mass m_1 is placed on a smooth horizontal table and the string passes over a small smooth pulley at the edge of the table such that the body of mass m_2 hangs vertically as in Fig. (38).

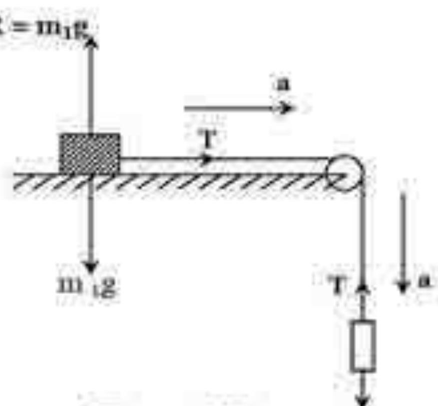


Figure (38)

We assume that the part of the string presented over the table is parallel to the table and perpendicular to is edge. The system is allowed to move, and it is required study the subsequent motion.

To study this problem, we consider separately the motion of each of the bodies of masses m_1 and m_2 in the vertical plane containing the pulley and the bodies in their initial position.

Motion of the Body of mass m_1 :

This body of mass is subject to three forces :

- The force of magnitude m_1g acting vertically downwards, where g is the magnitude of the constant acceleration of the earth's gravity.
- The reaction in the table which is vertically upwards since the table is smooth. Let R be its magnitude.
- The tension in the horizontal part of the string, acting towards the pulley, let its magnitude be T.

Since body of m_1 remains all the time on the table, therefore the sum of the vertical components of the forces acting on it must vanish.

$$R = m_1g \qquad (1)$$

This relation determines the magnitude of the table's reaction.

Now, the only horizontal force acting on body of mass m_1 is the tension T, thus it will require an acceleration in the direction of that force (i.e. towards the pulley), let "a" denotes its magnitude.

The equation of motion for the body of mass m_1 on the table is :

$$m_1a = T \qquad (2)$$

Motion of the body of mass m_2 :

Since the string remains tight all the time that hanging body of mass m_2 will have vertically downwards with an acceleration whose magnitude is also a.

The force acting on the body of mass m_2 are :

- The force of gravity of magnitude m_2g, vertically downwards.
- The tension in the vertical part of the string, of magnitude T along the vertical m upwards.

The equation of motion of the body of mass m_2 along the vertical is

$$m_2a = m_2g - T \qquad (3)$$

Adding (2) and (3)

$$(m_1 + m_2)\, a = m_2g$$

From which we get a

$$a = \frac{m_2g}{m_1 + m_2} \qquad (4)$$

The tension in the string may be found from equation (2), after substituting for "a" from equation (4).

<u>The pressure on the pulley :</u>

The string acts on the pulley by two forces, each of magnitude T, the first detected towards body of mass m_1 and other towards body of mass m_2. The resultant of these two forces represents the pressure excreted on the pulley, let P be its magnitude.

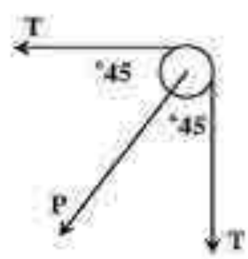


Figure (39)

Since the two forces of tension acting on the pulley, have equal magnitudes, therefore the pressure bisects the angle subtended by them, i.e. the pressure is it an inclination of 45° to the horizontal, its magnitude is given by :

$$P = 2T \cos \frac{90°}{2}$$

$$= 2T \cos 45°$$

$$\therefore P = \sqrt{2}\, T \qquad (5)$$

Example (1) :

A body of mass 195gm rests on a smooth horizontal table, and is connected by a string passing over a small smooth pulley at the edge of the table to a hanging body of mass 50gm. The system is left to move, starting from rest, when the first body is at a distance 100 cm from the pulley. Find the system's speed when the body of mass 195 gm reaches the pulley, find also the pressure on the pulley.

Solution :

Let the system have an acceleration of magnitude a as shown in Fig.(40),the magnitude of the acceleration of gravity is 980 cm/sec^2.

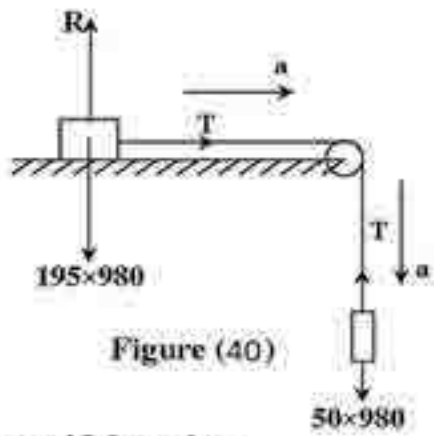


Figure (40)

The equation of motion for the body of mass 195 gm is :

$$195\, a = T \qquad (1)$$

The equation of motion for the body of mass 50 gm is :

$$50a = 50 \times 980 - T \qquad (2)$$

Adding (1) and (2)

$$245a = 50 \times 980$$

$$\therefore a = 200 \text{ cm/sec}^2$$

Suppose that V is the speed of the body on the table as it reaches the pulley.

$$V^2 = 2 \times 200 \times 100 = 40000$$

$$V = 200 \text{ cm/sec}$$

To determine the pressure on the pulley, we first obtain the tension in the string Substituting for a, in equation (1) we get

$$T = 195 \times 200 = 39000 \text{ dynes}$$

The pressure on the pulley is :

$$P = \sqrt{2}\, T = 39000\sqrt{2} \text{ dynes}$$

Example (2) :

A body of mass 200 gm is placed on a smooth horizontal table having two pulleys fixed at opposite edges, such that the body and the two pulleys lie on a straight line perpendicular to the table's edge. From two opposite points of the body, we fix two strings each passing over one of the pulleys. A body of mass 180gm is hanged from the first string, and a body of mass 110gm is hanged from the second. If the system starts moving from rest, determine the magnitude of its acceleration and the pressure on each of the pulleys.

Solution :

We expect the body of mass 180 gm to move vertically downwards ,let its acceleration be vertically downwards with magnitude a. Since the strings are tight all the time, the other two bodies will have acceleration of magnitude "a" for each, and whose directions are shown in Fig.(41). Let T_1, T_2 be the magnitudes of the tensions. The magnitude of the acceleration of gravity is $g = 980\ cm/sec^2$

The equation of motion for the body of mass 200gm placed on the table is :

$$200\,a = T_1 - T_2 \qquad (1)$$

The equation of motion for the body of mass 180gm is :

$$180\,a = 180 \times 980 - T_1 \qquad (2)$$

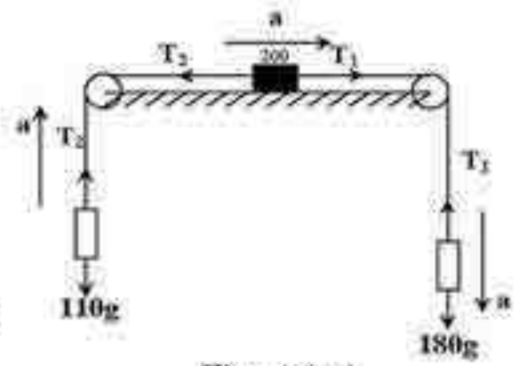


Figure (41)

The equation for the body of mass 110gm is :

$$110\,a = T_2 - 110 \times 980 \qquad (3)$$

Adding equation (1), (2) and (3) we get :

$$(200 + 180 + 110)\,a = (180 - 110) \times 980$$

$$a = \frac{180 - 110}{200 + 180 + 110} \times 980 = 140\ cm/sec^2$$

Since $a > 0$, our prediction that the larger mass will move vertically downward is correct .To get the pressure on each of pulleys, we must first find the tensions in the strings. Thus we substitute for "a" in equations (2) and (3).

$$T_1 = 180(980 - a) = 180(980 - 140) = 1.512 \times 10^5\ dynes$$

$$T_2 = 110(980 + a) = 110(980 + 140) = 1.232 \times 10^5\ dynes$$

The pressure on the first pulley :

$$P_1 = \sqrt{2}\,T_1 = \sqrt{2} \times 1.512 \times 10^5 = 151200\sqrt{2}\ dynes$$

The pressure on the second pulley :

$$P_2 = \sqrt{2}\,T_2 = \sqrt{2} \times 1.232 \times 10^5 = 123200\sqrt{2}\ dynes$$

Third application :

Motion of a system of two bodies, one of them is moving on a smooth inclined plane and the other vertically.

A particle of body of mass m_1 rests on the surface of θ smooth plane inclined at the angle a to the horizontal, and is connected by a string passing over a small smooth pulley at the top of the plane, to a body of mass m_2 hanging vertically beneath the pulley as shown in Fig.(42). Assuming that the part of the string on the plane is parallel to the line of greatest slope, if the system is left to move from rest, what is the subsequent motion.

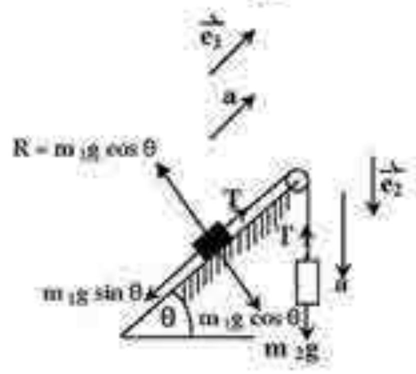


Figure (42)

We do not know before hang the direction of motion of the system, we thus consider two unit vectors $\vec{e_1}$, $\vec{e_2}$ the first along the line of greatest slope (say, upwards), and the second vertical (say, downwards) Let "a" be the algebraic measure of the acceleration of body of mass m_1 relative to $\vec{e_1}$. Since the string remains always tight, "a" is also the algebraic measure of the acceleration of body of mass m_2 relative to $\vec{e_2}$. This is consistent with the following.

If body of mass m_1 moves up the plane, body of mass m_2 will move vertically downwards ($a > 0$), and if body of mass m_1 moves down the plane body of mass m_2 will move vertically upwards ($a < 0$).

The motion of body of mass m_1 :

- Its weight of magnitude m_1g along the downwards vertical. We decompose this force into two components. One along the line of greatest slope towards the bottom of the plane, of magnitude $m_1g \sin \theta$, and the other normal to the plane and directed towards it of magnitude body of mass $m_1g \cos \theta$.
- The reaction of the plane. This is normal to the plane and directed away from it, let R be its magnitude.
- The tension in the string, acting along the line of greatest slope towards the top of the plane, let T be its magnitude.

Since the body of mass m_1 remains always on the inclined plane, the sum of the components of the forces normal to the plane must vanish.

$$R - m_1g \cos \theta = 0 \qquad (1)$$

Which determines the magnitude of the plane reaction.

The equation of motion of body of mass m_1 :

$$m_1a = T - m_1g \sin \theta \qquad (2)$$

The motion of body of mass m_2 :

This mass is subject to two forces.

- Its weight vertically downwards, of magnitude m_2g.
- The tension in the string vertically upwards of magnitude T.

The equation of motion of body of mass m_2 :

$$m_2g = m_2g - T \qquad (3)$$

adding equations (2) and (3)

$$(m_1 + m_2)\,a = (m_2 - m_1 \sin\theta)\,g$$

From which we get "a" :

$$a = \frac{m_1 - m_1 \sin\theta}{m_1 + m_2}\;g \qquad (4)$$

We may now obtain the tension in string from equation (3) after substituting for "a" from (4) We have three cases.

First :

If $m_2 > m_1 \sin\theta$, then $a > 0$

and the body of mass m_2 moves vertically downward where body of mass m_1 moves up the plane.

Second :

If $m_2 = m_1 \sin\theta$, then $a = 0$ and the system remains static (or both bodies move uniformly, with the same speed).

Third :

If $m_2 < m_1 \sin\theta$, then $a < 0$

and the body of mass m_2 moves vertically upwards, while body of mass m_1 moves down the plane.

The pressure on the pulley :

The string exerts two forces on the pulley, each of magnitude T, the first acting along the line of greatest slope downwards, and the second along the downwards vertical. The resultant of these two forces is the pressure on the pulley, let its magnitude be P. Since the two forces of tension acting on the pulley have the same magnitude, and subtend an angle whose measure is $(90° - \alpha)$, thus the pressure on the pulley bisects this angle, i.e. it is inclined to the downward vertical by an angle whose measure is $(45°- \frac{\theta}{2})$. The magnitude of the pressure is given by:

$$P = 2T \cos (45° - \frac{\theta}{2})$$

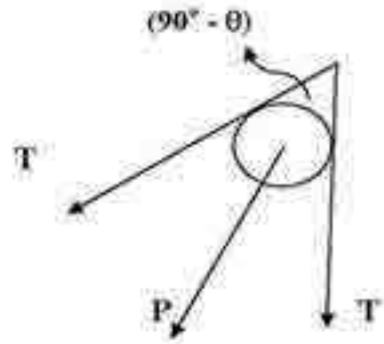


Figure (43)

Example (1) :

Two bodies of masses 10 , 8kg are connected by a string which passes over, a small smooth pulley at the top of a smooth plane inclined to the horizontal at 60°, while the other hanged vertically. If the system starts motion from rest, prove that each of the bodies moves with an acceleration of magnitude 0.36 m/sec^2 approximately. Show that the body of mass 8kg moves vertically upwards, and find the pressure on the pulley.

Solution :

Take a unit victor $\overrightarrow{e_1}$ parallel to the line of greatest slope pointing down the

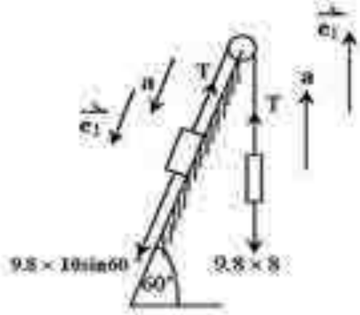


Figure (44)

plane, take also $\vec{e_2}$ a unit vector along the upward vertical. Let "a" be the algebraic measure relative to e_1 of the acceleration vector of the body of mass 10kgm. (a will also be the algebraic measure relative to $\vec{e_2}$ of the acceleration vector of the body of mass 8kg) as shown in Fig. (44).

The acceleration of gravity has a magnitude $g = 9.8$ m/sec^2

The equation of motion for the body of mass 10kg :

$$10\,a = 10 \times 9.8 \times \frac{\sqrt{3}}{2} - T \qquad (1)$$

The equation of motion for the body of mass 8kg :

$$8\,a = T - 8 \times 9.8 \qquad (2)$$

adding (1) and (2) we get a :

$$18\,a = 10 \times 9.8 \times \frac{\sqrt{3}}{2} - 8 \times 9.8$$

$$a \simeq 0.36 \text{ m/sec}^2$$

We observe that "a" turned out to be positive, which means that the directions which we specified for the accelerations of the bodies are actually as shown in the figure. Thus the body of mass 8kg moves vertically upwards. Substituting for a in equation (2), we get the magnitude of the tension in the string.

$$8 \times 0.36 \simeq T - 8 \times 9.8$$

$$T \simeq 8 \times 0.36 + 8 \times 9.8$$

$$\simeq 81.28 \text{ newtons}$$

The pressure on the pulley makes, with the downward vertical, an angle of measure.

$45^\circ - 30^\circ = 15^\circ$, its magnitude is give, by

$P \simeq 2 \times 81.28 \times \cos 15^\circ$

$P \simeq 157$ newtons

Example (2) :

Two bodies of masses 5 and 4kg, are connected by a string. The large body is placed on a smooth plane inclined to the horizontal by an angle of measure α, where $\tan \alpha = \frac{3}{4}$. The string passes over a small smooth pulley at the top of the plane, so that the other body hangs vertically below the pulley as shown in Fig.(45).

If the system is allowed to move, find the magnitude of its acceleration, and the tension in the string. If the string is cut after one second from the start of the motion, find the distance ascended by the body of mass 5kg up to plane from the moment the string was cut until it rests instantaneously :

Solution :

Let $\overrightarrow{e_1}$ be a unit vector parallel to the line of greatest slope and directed up the plane, and $\overrightarrow{e_2}$ be a unit vector along the downward vertical.

Assume that a is the algebraic measure of the acceleration of the bodies of masses 5 and 4 kg relative to $\overrightarrow{e_1}$ and $\overrightarrow{e_2}$ respectively. We take for the magnitude of the earth's acceleration $g = 9.8$ m/sec^2.

The equation of motion the body of mass 5kg :

$$5\,a = T - 5 \times 9.8 \times \frac{3}{5} \qquad (1)$$

The equation of motion the body of mass 4kg :

$$4\,a = 4 \times 9.8 - T \qquad (2)$$

Figure (45)

adding (1) and (2) we get :

$$9\,a = 9.8$$

$$a = \frac{49}{45}\ \text{m/sec}^2$$

This means that the body of mass 5kg moves up the plane, while the body of mass 4kg moves vertically downwards.

We substitute for "a" in equation (2), to obtain the tension.

$$4 \times \frac{49}{45} = 4 \times 9.8 - T$$

$$T = 4 \times 9.8 - \frac{4\times 49}{45} = \frac{1568}{45}\ \text{newtons}$$

After one second from the commencement of motion, the speed of body on the plane is :

$$V = a \times 1 = \frac{49}{45}\ \text{m/sec}$$

When the string is cut the body of mass 5kg moves under the action of the component of its weight parallel to the line of greatest slope, which is directed down the plane. Thus it moves with a retarding acceleration whose algebraic measure a_1, relative to $\overrightarrow{e_1}$ is given by the relation :

$$a_1 = -\,5.88\ \text{m/sec}^2$$

This acceleration will cause the body to slow until it comes to rest instantaneously. The distance S traversed during this motion is given by

$$V^2 - V_0^{\,2} = 2a_1\, S \qquad \text{where} \qquad V = 0\ ,\ V_0 = \frac{49}{45}\ \text{m/sec}$$

$$-\left(\frac{49}{45}\right)^2 = 2 \times 5.88\ S$$

$$S = \frac{49}{486}\ \text{m.} \simeq 0.1\text{m} \qquad \text{or} \qquad S \simeq 10\ \text{cm}$$

Exercises (2 - 1)

1- A string passing round a small smooth pulley carries at its free ends, bodies of masses $\frac{1}{2}$ and $\frac{1}{5}$ kg. Determine the acceleration of the system and the pressure on the pulley.

2- Two bodies of masses 125 and 120 gm hang from the ends of a string which passes over a small smooth pulley. Find their common acceleration and the pressure on the pulley. If the motion starts when the two bodies are on the same horizontal level, what is the vertical distance between them after one second.

3- Two bodies of masses of 4 and 4.1kg are connected by a string of length 150cm, passing over a small smooth pulley, and the two parts of the string are vertical. Prove that the resulting acceleration has a magnitude of $12 cm/sec^2$ approximately and if the motion starts from rest when the body of large mass is at the pulley, what is the speed of the body of small mass when it reaches the pulley.

4- A string passing round a smooth pulley, holds at its free end, a body of mass 4kg, and at the other end two bodies of masses 3kg and 2kg, are attached. If the system starts moving from rest, find its acceleration, and the speed acquired by the body of mass 4kg, after 3 seconds. If at this moment, the body of mass 2kg is removed, show that a time of $\frac{7}{3}$ sec will elapse before the body of mass 4kg will first come to rest.

5- Two bodies of masses 4 and 10kg are connected by a string passing over a small smooth pulley. Find the common acceleration and the pressure on the pulley. If $\frac{2}{5}$ of the body of large mass is separated from the system, show that the pressure on the pulley is $\frac{84}{100}$ of its previous value.

6- Two bodies of masses 20 and 960gm are connected by a string passing over a smooth pulley at the table's edge. The body of larger mass is placed on the table, while the small mass hangs vertically below the pulley, such that the horizontal part of the string is perpendicular to the table's edge. Find :

i) the system's acceleration.

ii) The tension is the string.

iii) The pressure on the pulley.

7- A body of mass 3kg placed on a smooth horizontal table, is connected by a string passing over a pulley at the table's edge to a body of mass 0.675kg. The horizontal part of the string is perpendicular to the table's edge. Find the acceleration of the system. If the motion starts from rest, when the body of larger mass is at a distance of 250cm from the pulley, find its speed when just about to hit the pulley.

8- A body of mass 100gm is placed on smooth horizontal table from two opposite points in the body we fix two strings each of them passing over a pulley at the table's edge, such that the body and the pulleys lie on a straight line perpendicular to the table's edge. Two bodies of masses 300 and 350gm hang from the free ends of the strings. Determine the magnitude of the system's acceleration, and the tension in each string.

9- Two string are fixed to a body of mass 500gm from two opposite points of it. The body is placed on a smooth horizontal table having two pulleys A, B fixed at the table's opposite edges such that the body and pulley lie on a straight line perpendicular to the edge of the table. Each of the string passes over pulleys. From the free ends of the string passing over pulleys A, B respectively of two bodies of mass 300 , 200gm.Hang vertically below the pulley. The system is left to move string from rest. The body on the table is at a distance 245cm from the pulley A. After one second from the commencement of motion one third of the body of mass 300gm is separated. Show that the body of mass 500gm bits the pulley A after two more seconds.

10- Two bodies of equal masses 2kg each are connected by a string. One of the bodies is placed on a smooth plane inclined to the horizontal by an angle whose measure is 30°, and the string passes over a small smooth pulley at the top of the plane, such that the other body hang vertically below the pulley. Find the acceleration of the system, and the tension in the string. Find also the pressure on the pulley.

11- Two bodies of masses 60, 20gm, are connected by a string which passes round a small smooth pulleys fixed at the top of smooth plane inclined by an angle of measure 30° with the horizontal. The first body rests on the plane while the second hangs vertically below the pulley. Determine the acceleration of the system, the tension in the string, and the pressure on the pulley.

12- A body of mass 60gm is tied to a string and placed on a smooth plane which makes an angle of measure 30° with the horizontal. The string passes over a small smooth pulley at the top of the plane, and holds a body of mass 40gm vertically below the pulley, find the system's acceleration. If the system starts its motion from rest when the body on the plane is at a distance 196cm from the pulley, when does the body reach the pulley?

13- Two bodies of masses 4, 3kg are connected by a string which passes over a small smooth pulley fixed at the top of a plane at an inclination of measure 30° with the horizontal. The first body is placed on the plane, while the second hangs vertically below the pulley. Find the system's acceleration and the pressure on the pulley, and if the system starts moving from rest and the string is cut after 3 second from the start of motion, what is the distance traversed by the body on the plane before it comes to rest instantaneously?

14- Two bodies x and y of mass 132 and 108gm respectively are tied to the two ends of a thread passing through a smooth vertical pulley. The body (y) is tied to another thread of length 60cm and carries at its other end a body (z) of body of mass 90gm which hangs vertically. The system began its motion when the body (z) was at a height 12.5cm above the ground. Prove that the body (y) rests instantaneously when it is 35cm above the ground.

15- Two bodies A and B, each of mass m gm are tied to the two ends of a thread passing through a smooth vertical pulley so that they hang vertically. A body of mass 35gm is added to the body A and the system began its motion from rest. Prove that the acceleration of the system is $\frac{35g}{2m+35}$ cm/sec^2, where g is the acceleration of gravity. If the body A hits the ground after witting a distance 50cm and the body B continues its motion till it is 60cm above it starting position and comes to rest, find the value of m.

Motion on a rough plane

Under the heading " Applications on Newton's laws ", we considered the motion of a system of two bodies connected by a string when either of the bodies or both of them move on a smooth plane. The motion of this system will be considered now where the plane is assumed rough, consequently a new force will appear in the equations of motion, namely, the friction between the body and the plane.

In the section on static's, we have dealt in detail with friction. Here, we shall only give the basic rules which have to be observed when dealing with motion on a rough plane :

1- The friction is always directed against the direction of actual or possible motion.

2- As the force tending to cause motion increases, until it reaches a maximum value which can not be surpassed. In which case, the body is just about to move and the friction exerted is called limiting friction.

3- When motion occurs, the friction is the limiting friction. If μ denotes the coefficient of friction, R denotes the magnitude of the normal reaction, f denotes the magnitude of limiting friction, we have :

$$f = \mu R$$

Example (1) :

A body of mass 3kg placed on a rough horizontal plane is connected by a string which passes round a smooth pulley at the plane's edge, to a mass of 2kg. If the coefficient of friction is $\frac{1}{3}$, find the acceleration of the system and the distance traversed in one second.

Solution :

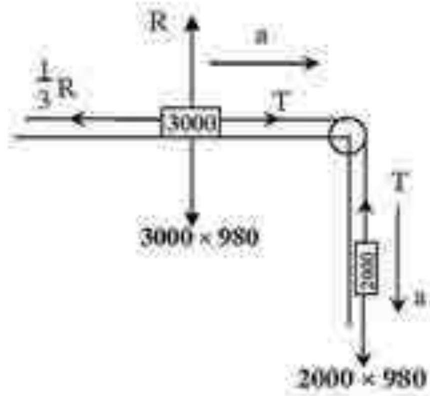


Figure. (46)

The forces acting on each of the bodies shown in Fig. (46).

Let the magnitude of the system's acceleration be " a ", since the body on the plane has no vertical motion, it is clear that the vertical forces exerted on it are balanced.

$$R = 3000 \times 980 \text{ dynes}$$

Since the body on the plane is in motion, therefore the friction is the limiting friction.

$$F = \mu R$$

$$= \frac{1}{3} \times 3000 \times 980 \text{ dynes}$$

The equation of motion for the body on the plane is :

$$3000a = T - \frac{1}{3} \times 3000 \times 980$$

The equation of motion for the mass 2kg is :

$$2000 \times a = 2000 \times 980 - T$$

Adding equations (1) and (2) :

$$\therefore 5000a = 2000 \times 980 - 1000 \times 980$$

$$\therefore a = 196 \text{ cm/sec}^2$$

To get the distance required, we use the formula

$$S = V_0 t + \frac{1}{2} a t^2$$

$$= \frac{1}{2} \times 196 \times 1 = 98 \text{ cm}$$

Example (2) :

In the previous, if the system starts motion forces rest and the hanged body is at a height 50 cm above the ground, find the distance traversed by the body on the plane before it comes to rest.

Solution :

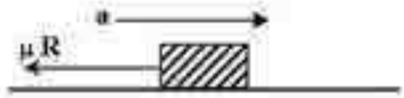


Figure. (47)

We first obtain the speed acquired by the system when the hanging body reaches the ground, from the formula.

$$V^2 = V_0^2 + 2\,a\,S$$

$$\therefore V^2 = 0 + 2 \times 196 \times 50$$

$$\therefore V = 140 \text{ cm/sec}^2$$

As the hanged body reaches the ground it rests on it, and the body on the plane becomes subject to the friction only. This is shown in Fig. (47). This causes it to decelerate, its equation of motion being.

$$3000a = \frac{-1}{3} \times 3000 \times 980$$

$$\therefore a = \frac{-980}{3} \text{ cm/sec}^2$$

But the initial velocity of the body for this motion is 140 cm/sec so to get the distance traversed by the body until it comes to rest we use

$$V^2 = V_0^2 + 2\,a\,S$$

$$0 = (140)^2 - 2 \times \frac{980}{3} \times S$$

$$\therefore S = 30 \text{ cm}$$

Example (3) :

A body is placed on the top of a rough inclined plane of length 250cm. and height 150cm. The body starts sliding down the plane, if the coefficient of friction is $\frac{1}{2}$, find the acceleration of the body ,its speed after it has moved 250 cm on the plane , and if the body be projected from the lowest point, find the minimum speed of projection so that the body reaches the highest point.

Solution :

$$MA = 250\,cm\ ,\ BA = 150\,cm\ ,\ MB = 200\,cm$$

$$\because\ \sin\alpha = \frac{150}{250} = 0.6 \qquad ,\qquad \cos\alpha = \frac{200}{250} = 0.8$$

The body slides downwards, thus the friction f acts upwards and is equal to μ R, where R is the normal reaction Fig. (15). Resolving along the direction of greatest slope in the plane.

$$m\,a = m\,g\,\sin\alpha - \mu R \qquad (1)$$

resolving along the normal to the plane.

$$R = m\,g\,\cos\alpha \qquad (2)$$

(the acceleration in the direction of the normal to the plane is zero).

Substituting for R from (2) in (1), we get

$$m\,a = m\,g\,\sin\alpha - \mu\,m\,g\,\cos\alpha \qquad (3)$$

$$\therefore\ a = g\,(\sin\alpha - \mu\cos\alpha) \qquad (4)$$

$$\therefore\ a = 980\,(0.6 - \frac{1}{2}\times 0.8) = 196\ cm/sec^2$$

To obtain the speed of the body after travelling 200 cm, we use :

$$V^2 = 2\ a\ S$$

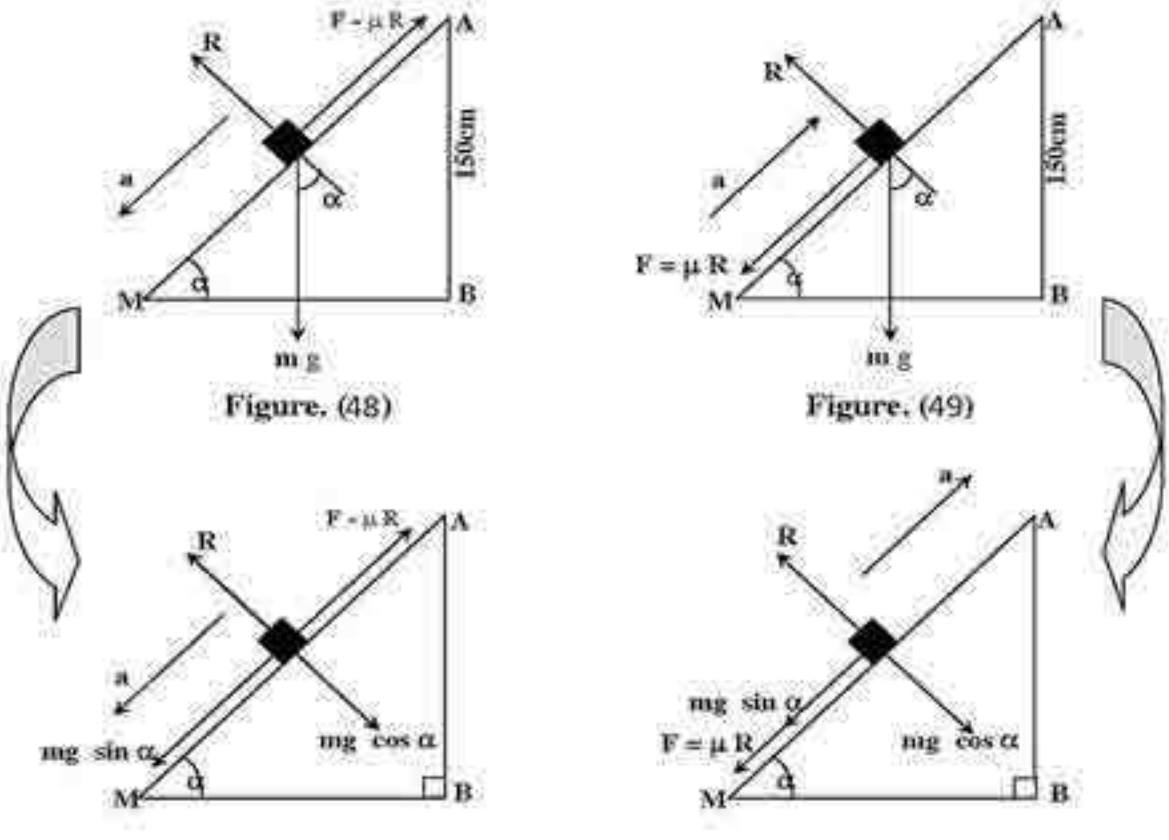


Figure. (48) Figure. (49)

$$V^2 = 2 \times 196 \times 200$$

$$V = 280 \text{ cm/sec}$$

Second :

Since the body moves up the plane, therefore, the friction is down the plane, Fig. (49). Resolving along the plane and the normal to it, we have :

$$m\ a = -\ m\ g \sin\alpha - \mu R$$

$$R = m\ g \cos\alpha$$

$$m\ a = -\ m\ g \sin\alpha - \frac{1}{2} m\ g \cos\alpha \qquad (5)$$

$$a = -\ g\ (\sin\alpha + \mu \cos\alpha)$$

$$a = -980\left(0.6 + \frac{1}{2} \times 0.8\right)$$

$$a = -980 \text{ cm/sec}^2$$

To get the minimum speed of projection, which allows the body to reach the top we set $V = 0$, $S = 250$ cm and $a = -980$ cm/sec^2 in the relation.

$$V^2 = V_0^2 + 2\,a\,S$$

$$0 = V_0^2 - 2 \times 980 \times 250$$

$$V_0^2 = 1960 \times 250$$

$$V_0 = 700 \text{ cm/sec}$$

Example (4) :

A body of mass 10gm rests on a rough plane inclined by an angle of measure 30 to the horizontal. The body is connected by a string which passes round a small smooth pulley fixed at the plane's top, to a body of mass 15gm which hangs vertically. If the coefficient of friction is $\frac{1}{\sqrt{3}}$, find the time which passes before the first body moves a distance 100cm on the plane, and obtain its speed then.

Solution :

Referring to Fig. (50) the equations of motion of the mass 10gm along the plane and the normal to the plane are :

$$10\,a = T - 10 \times g \sin 30 - \mu R \qquad (1)$$

$$R = 10g \cos 30° \qquad (2)$$

The equation of motion for the second body is :

$$15\,a = 15\,g - T \qquad (3)$$

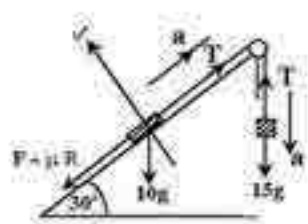


Figure. (50)

Substituting for R from (2) in (1)

$$10\,a = T - 10\,g \times \frac{1}{2} - \frac{1}{\sqrt{3}} \times 10\,g \times \frac{\sqrt{3}}{2}$$

$$10\,a = T - 10\,g$$

Adding (3) and (4)

$$25\,a = 5\,g$$

$$a = \frac{980}{5} = 196\ \text{cm/sec}^2$$

To get the time required to traverse a distance of 100cm starting from rest, with an acceleration 196cm/sec^2, we use the formula :

$$s = \frac{1}{2}\ a\,t^2$$

$$100 = \frac{1}{2} \times 196 \times t^2$$

$$t^2 = \frac{200}{196}$$

$$t = \frac{5\sqrt{2}}{7}\ \text{sec}$$

To obtain the speed we use :

$$V^2 = 2\,a\,S$$

$$= 2 \times 196 \times 100$$

$$V = 140\sqrt{2}\ \text{cm/sec}$$

Exercises (2 - 2)

1- A body of mass 60 gm placed on a horizontal table is connected by a string which passed over a small smooth pulley at the table's edge, to mass 30gm hanging vertically. Find the acceleration of the system, where coefficient of friction is 0.5.

2- Two bodies of masses 50 and 30gm are tied to the ends of a string which passes round a smooth pulley fixed at the edge of a horizontal plane. The first body rests on the plane, and the second hangs vertically, find the acceleration and the tension in the string, given that the coefficient of friction 0.2.

3- A body of mass 40gm resting on a horizontal plane is connected, by a string which passes round a smooth pulley at the plane's edge, to a body of mass 30gm which hangs vertically. If the coefficient of friction is 0.5, find the acceleration of the system and the distance traversed after 7 seconds from the start of motion.

4- Two equal bodies of masses 40gm each are connected by a string which passes over a smooth pulley fixed at the edge of a horizontal table. One of the bodies is placed on the table while the other hangs vertically. The motion starts from rest when the hanging body is at a height of 10cm above the ground. Find the distance traversed by the body on the table before it comes to rest, where the coefficient of friction is 0.5.

5- In a factory, boxes are transmitted by sliding them on an inclined plane of length 15m and height 9m. If a box starts from rest at the top of the plane, find its speed when it reaches the bottom of the plane, (i) when the plane is smooth, (ii) when the plane is rough, the coefficient of frication being 0.25.

6- A body slides from the top of an inclined plane of length 4.5 m. and height 2.7 m stating from rest. Determine is speed as it reaches the bottom of the plane, given that the coefficient of friction is 0.5.

7- A body slides on a rough plane inclined to the horizontal by an angle of measure 45°. If the coefficient of friction is 0.75, show that the time required to traverse a given distance is twice the time required to traverse the same distance, had the plane been smooth.

8- A body of mass 50gm is placed on a rough horizontal table the coefficient of friction being $\frac{1}{5}$. The body is connected by a string passing over a smooth pulley at the table's edge, to a body of mass 30cm. Find the system's acceleration.

9- Two equal bodies of masses 20gm are connected by a string which passes over a pulley at the edge of a rough horizontal table. One of the bodies is placed on the table at a distance 15cm from the pulley, the other hangs vertically at a height of 10cm above the ground. If the coefficient of friction is $\frac{1}{2}$, and the system starts motion from rest, prove that the acceleration of the system is 245 cm/sec^2.

Find the speed when the hanged body reaches the ground, and the distance traversed by the other body after that, before it comes to rest.

10- A body of mass 120gm rests on a rough plane inclined to the horizontal by an angle whose tangent is $\frac{3}{4}$. The body is connected, by a string which passes over a smooth pulley fixed at the top of the plane, to a body of mass 160gm. If the coefficient of friction is $\frac{2}{3}$, and the system starts its motion from rest, determine the distance traversed in three seconds.

11- A body of mass 350gm is placed on a rough planed inclined to the horizontal at an angle whose tangent is $\frac{3}{4}$. The body is tied to one end of light string which passes through a smooth pulley fixed at the top of the plane. The other end of the string carries a balance scale of body of mass 70gm. If the coefficient of friction between the body and the plane is $\frac{1}{4}$, find the least mass of the body that must be put on the scale so that the body remains at rest. If an additional body of mass 280gm is put on the scale, find the acceleration of motion and the tension in the string.

12- A body of mass 200gm is placed on a rough horizontal plane and tied to one end of a light string passing though a smooth pulley fixed at the end of the table. The other end of the string carries a 200gm body which hangs vertically. The system starts its motion from rest when the string was tight, and the hanging body is 90cm above the ground, and the body on the table, is 135cm from the pulley. If the coefficient of friction between the body and the plane is $\frac{1}{2}$, prove that the system moves with an acceleration of 145 cm/sec^2 and find the velocity of the system when the hanging body hits the ground. Would the body on the table reach the pulley? Explain your answer.

13- A rough plane is inclined to the horizontal at an angle $30°$ and is joined at its top to another rough horizontal plane. A body of mass 60gm is placed on the horizontal plane and joined to one end of a string which passes through a smooth pulley fixed at the common edge of the two planes. A body of mass of 100gm is joined to the other end of the string and is placed on the inclined plane. If both branches of the string are perpendicular to the common edge find the acceleration of the system and the tension in the string given that the coefficient of friction between the first body and the horizontal plane is $\frac{1}{4}$ and between the body and the inclined second plane is $\frac{1}{2\sqrt{3}}$. If the string is cut after 4 seconds from the start of motion find the total distance that the body of mass 60 gm moves before coming to rest.

14- A body of mass 250gm is placed on a rough plane inclined to the horizontal at an angle whose tangent is $\frac{4}{3}$. The body is joined to one end of a string passing through a pulley fixed at the top of the plane. At the other end of the string, a body of mass k gm hangs vertically. If the least value of k necessary to keep the body at the rest on the plane is 150gm, prove that the coefficient of friction between the body and the plane is $\frac{1}{3}$. If a body of mass 350gm is hanged from the free end of the string, find the acceleration of motion of the system.

Chapter Three

Impulse and Collision

Preface :

In this chapter we will deal with: The effect of a fixed forces on a particle for an infinitesimal time interval and the study of the impact of smooth spheres .

Objectives:

By the end of teaching this chapter, the student should be able to:

(1) Recognize the concept of impulse .

(2) Recognize the impulsive force.

(3) Recognize the relation between impulse and the change of momentum.

(4) Recognize that sum of momentum of two bodies before impact = their sum after impact.

Topics :

(1) Impulse .

(2) Impulsive force.

(3) Collision .

Impulse and Collision

Introduction :

In many cases we have to deal with some kinds of motion in which the velocity vector of a body changes substantially either in magnitude or direction or both during a very short time, as it happens in many phenomena or our daily life, such as the impact of the wheels of aeroplanes on the runways when landing, the coupling of a train engine with the trucks, collision of cars, etc...

In such cases, the study of the motion of a body during these very short intervals is very difficult, and in fact impossible, due to the interference of many factors such as the change in the shape of the colliding bodies, the mutual heat effects, etc.

To overcome these difficulties, we can use the experimental and laboratory laws which give us some information about what happens during this short interval of time, in which it is difficult to study the motion and using this information to study the motion, and to find a relation between the state of the body before and after this drastic change in the velocity vector.

Thus, it is useful to introduce the concept of "Impulse" and "Impulsive forces" and that is what we are going to discuss in what follows.

Impulse :

If a constant force $\overrightarrow{F}$ acts on a particle of constant mass during a time interval t, we define the impulse of this force, denoted by $\overline{I}$ as the product of the force vector and the time of its action.

$$\overrightarrow{I} = \overrightarrow{F}\ t \qquad (1)$$

It is clear from this definition that the impulse is a vector in the same direction as the force vector.

We can also write the following relation between the magnitude of the impulse vector and the magnitude of the force vector.

$$I = F\,t \qquad (2)$$

If the magnitude of the force is finite, the magnitude of its impulse tends to zero when the time interval of action tends to zero, as it is clear from (2). In our study we will be concerned only with rectilinear motion under the action of a constant force parallel to this line.

Theorem :

If a constant force acts on a particle for an infinitesimal interval of time, the change in the momentum during this interval is equal to the impulse of the force.

Proof :

Consider two positions of the particle at two very close instants of time, t, t + h, let H, H' be the algebraic measures of its momentum at these two instants respectively relative to a unit vector parallel to the straight line on which motion occurs.

From Newton's second law : $\frac{dH}{dt} = F$

where F is the algebraic measure of the force.

since $\frac{dH}{dt} = \lim_{h \to 0} \frac{H' - H}{h}$

we can write $\frac{dH}{dt} = \frac{H' - H}{h}$ approximately

$\therefore \frac{H' - H}{h} = F$

i.e. $H' - H = F\,h$

but the right hand side of this relation is equal to the impulse of the force during the interval of time h.

$$\therefore \; I = H' - H \qquad (1)$$

If m is the mass of the particle is constant v, v' be the algebraic measures of the velocity vectors at the instants t, t + h respectively we can write the last relation in the form.

$$I = m\,v' - m\,v \qquad (2)$$

Impulsive Forces :

Consider now a special kind of the forces which is characterized by the following property : "When the interval of action of the force decreases the magnitude of the force increases so that the following relation is satisfied.

$$\lim_{t \to 0} (\overrightarrow{F}\, t) = \text{a finite non-zero number.}$$

The force in this case is called an impulsive force its impulse is defined as follows :

$$\overrightarrow{I} = \lim_{t \to 0} (\overrightarrow{F}\, t) \neq \overrightarrow{0}$$

It's magnitude

$$I = \lim_{t \to 0} (F\, t) \neq 0$$

And thus an impulsive force ;

1. acts on the body during an infinitestimal interval of time.
2. its magnitude is infinitely great.
3. the magnitude of its impulse is finite and not equal to zero (or its impulse vecotr is of finite magnitude, and is not equal to the zero vector).

It can be proved that relation (1) is still true in case of impulsive forces.

Units of measurements of the magnitude of impulse :

From the definition of impulse we have

Unit of measuring magnitude of impulse

= unit of measuring magnitude of force × unit of measuring time.

and thus the units of measuring magnitude of impulse, may be : newton. second, or dyne. second, etc...........

Also, we can express units of measuring magnitude of impulse in another way by noticing that

Magnitude of impulse = magnitude of change in momentum.

∴ Unit of measuring magnitude of impulse.

= unit of measuring magnitude of mass × unit of measuring magnitude of velocity and thus the unit of measuring magnitude of impulse may be :

kilogram. metre/second or gram. centimetre/second etc. If we adopt the above table of units, then :

1. If the mass is measured in kilogram, magnitude of velocity in metre per second, the unit of magnitude of impulse will be kg.m/sec or newton. second.
2. If the mass is measured in gram, magnitude of velocity in centimetre per second, the unit of magnitude of impulse will be gm.m/sec or dyne. second.

Impact :

Impact of smooth spheres :

If two spheres collide, so that we can consider this collision instantaneous, (i.e. it lasts a very short time) and assuming that the two spheres keep their shape, then the impact between them occurs at a single point which is their point of contact at the instant of collision.

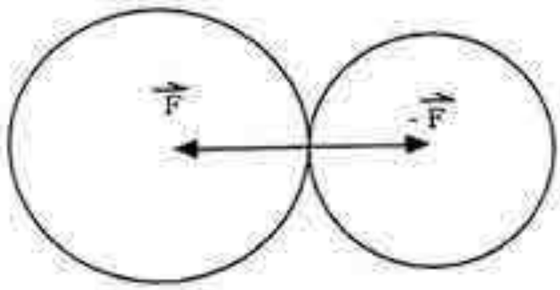


Fig. (51)

At this instant each sphere acts on the other with a certain force so that these two forces satisfy Newton's third law, i.e. they are equal in magnitude and in opposite directions. It is natural to believe that the mutual forces between the two spheres are impulsive forces. Since we notice a substantial change in the velocities of the spheres just before and after impact during the infinitesimal time of impact.

Thus the impulse of the first sphere on the second is equal in magnitude and opposite in direction to the impulse of the second sphere on the first.

It was found experimentally that if the two spheres are smooth, the force with which each sphere acts on the other will be along the line of centres at the instant of impact.

If the velocities just before impact are parallel to the line of centres at the instant of impact, the impact is called direct impact, otherwise it is called indirect or oblique impact. This last kind of impact is outside the scope of this book.

In what follows we are going to consider the direct impact of two smooth spheres.

Let m_1, m_2 be the masses of the two spheres, $\overrightarrow{I}$ the impulse of the second sphere on the first. $\overrightarrow{v_1}$, $\overrightarrow{v_2}$ the velocity vectors of the two spheres just before impact, $\overrightarrow{v_1'}$, $\overrightarrow{v_2'}$ their velocity vectors just after impact.

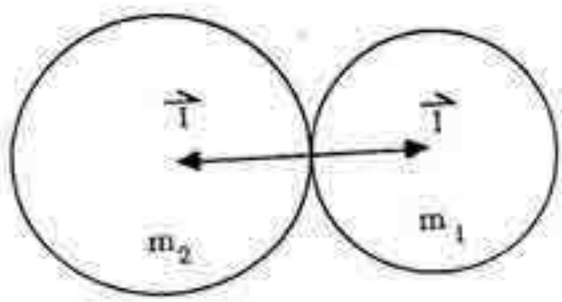


Fig. (52)

$\therefore$ the change in momentum of each sphere

= the impulse acting on it.

For the first sphere :

$$m_1 \overrightarrow{v_1'} - m_1 \overrightarrow{v_1} = \overrightarrow{I}$$

but the two vectors $\overrightarrow{v_1}$, $\overrightarrow{I}$ are parallel to the line of centres, therefore the vector $\overrightarrow{v_1'}$ is also parallel to this line.

For the second sphere :

$$m_2 \overrightarrow{v_2} - m_2 \overrightarrow{v_2} = - \overrightarrow{I}$$

Since each of the vectors $\overrightarrow{v_2}$, $- \overrightarrow{I}$ is parallel to the line of centres, therefore $\overrightarrow{v_2'}$ is also parallel to this line.

Thus we reach the following important result :

"In direct impact the velocities just after impact are parallel to the lines of centres".

Adding the last two relations, we get :

$$(m_1 \overrightarrow{v_1} - m_1 \overrightarrow{v_1}) + (m_2 \overrightarrow{v_2} - m_2 \overrightarrow{v_2}) = \overrightarrow{0}$$

$$\therefore \ m_1 \overrightarrow{v_1} + m_2 \overrightarrow{v_2} = m_1 \overrightarrow{v_1} + m_2 \overrightarrow{v_2}$$

and thus the following theorem is satisfied :

Theorem :

If two smooth spheres collide, then the sum of their momentum does not change as a result of impact . Since the velocities before and after impact are parallel to the direction of the line of centres at the instant of impact, we can use the algebraic measures of the velocity vectors, and impulse instead of the vectors itself.

Consider the straight line coincident with the line of centres at the instant of impact, and choose a unit vector $\hat{c}$ determining the positive direction Fig. (53).

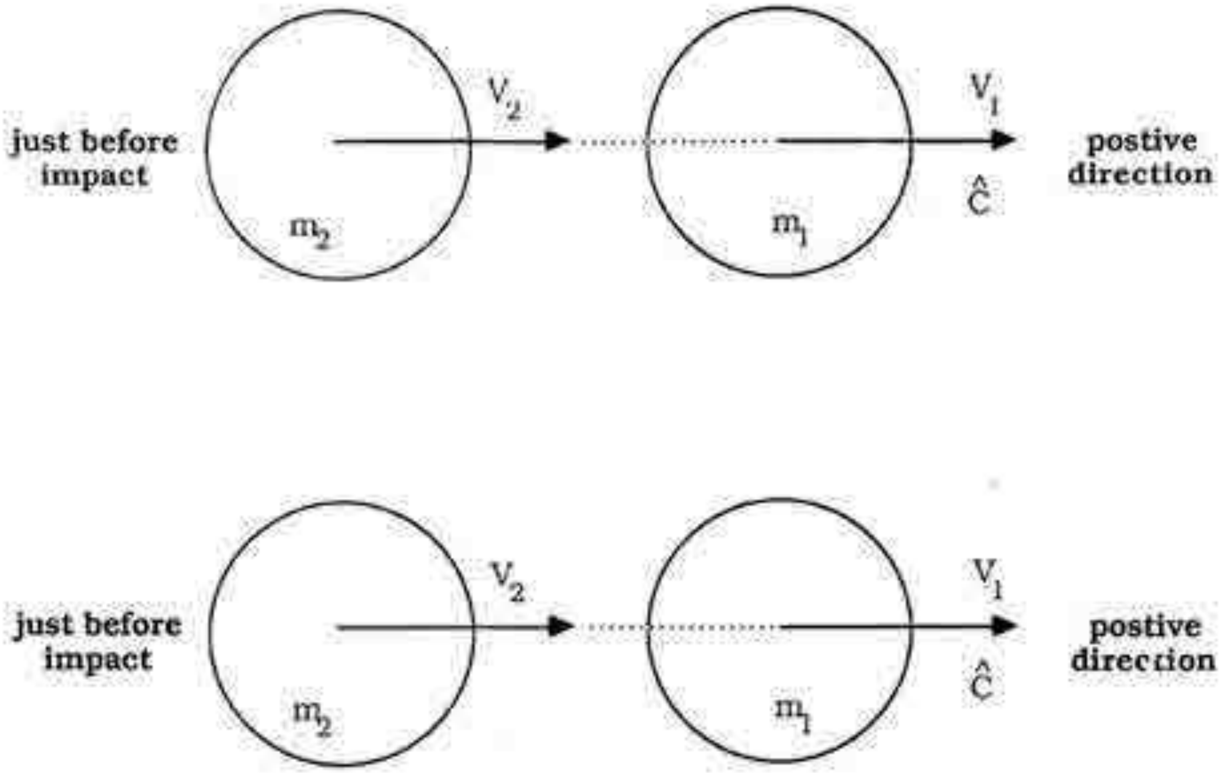


Fig. (53)

Let I be the algebraic measure of the impulse of the second sphere on the first one, v_1 , v_2 the algebraic measures of their velocities just before impact and, $v_1^{'}$, $v_2^{'}$ the algebraic measures of their velocities just after impact, the above relations take the following forms :

$$m_1 v_1' - m_1 v_1 = I \qquad (1)$$

$$m_2 v_2' - m_2 v_2 = -I \qquad (2)$$

$$m_1 v_1' + m_2 v_2' = m_1 v_1 + m_2 v_2 \qquad (3)$$

Example (1) :

A smooth sphere of mass 200 gm moves in a straight lien on a horizontal ground with velocity 10m/sec. If this sphere impinges on a smooth vertical wall normal to the direction of its velocity, and it rebounds from it with a velocity 2 m/sec. Find the magnitude of the impulse of the wall on the sphere.

Solution :

Consider the positive direction on the straight line on which the motion occurs as shown in Fig. (54). Consider the wall as a smooth sphere of a very large radius (and of a very large mass)

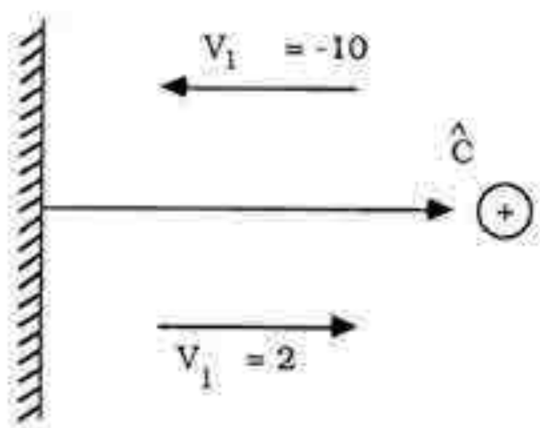


Fig. (54)

$v_1 = -10$ m/sec $\quad v_2 = 2$ m/sec $\quad m_1 = 200$ gm.

Applying relation (2)

$200 \times 2 - 200 \times (-10) = I$

$\therefore \quad I = 200 \times (2 + 10) = 200 \times 12 = 2400$ gm. m/sec.

notice that the units of impulse are not uniform (although it is true) since it contains the gram and metre together. If we want to express the magnitude of the impulse in uniform units we have to transfer to unit of kilogram (M.K.S. system) or change the magnitude of velocity to unit of cm/sec. (C.G.S.) system).

In the first case put $m_1 = 0.2$ kg.

$\therefore \; I = 0.2 \times 12 = 2.4$ kg. m/sec.

$= 2.4$ newtons. sec

In the second case put $v_1 = 1000$ cm/sec. $v_1 = 200$ cm/sec.

$\therefore \; I = 200 \times 1200$

$= 2.4 \times 10^5$ gm.cm/sec.

$= 2.4 \times 10^5$ dyne sec.

Example (2) :

Two smooth spheres each of mass 200 gm are moving in the same straight line on a horizontal ground the first with velocity 5 m/sec. and the second with velocity 9 m/sec. in the same direction as the first. If the two spheres collide, find the velocit of each just after impact. given that the magnitude of the impulse of the second sphere on the first is equal to 0.6×10^5 dynes.

Solution :

Let the positive direction be that the velocity vectors just before impact Fig. (55)

$\therefore \quad v_1 = 5$ m/sec.

$v_2 = 9$ m/sec.

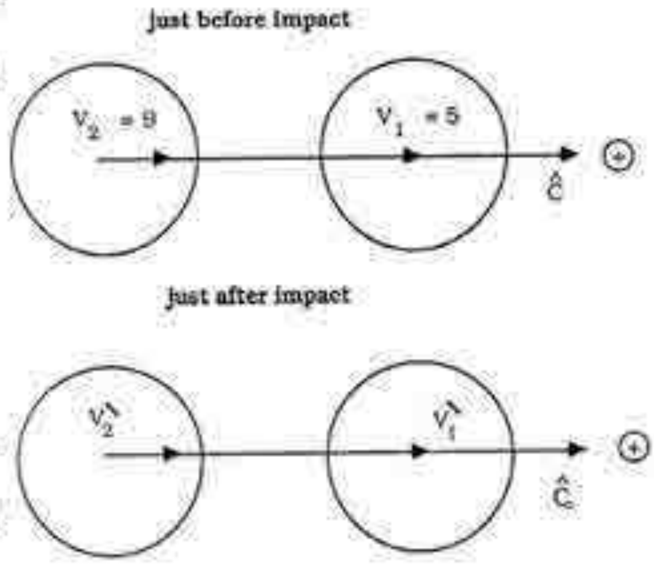


Fig. 55

Since the impulse of the second sphere on the first sphere is in the positive direction.

$\therefore \quad I = 0.6 \times 10^5$ dyne. sec

Since the units of velocity and impulse are not uniform, we have to change the veocities to units of cm/sec. or the magnitude of impulse to units of newton. second.

Choosing the first method $v_1 = 500$ cm/sec.

$v_2 = 900$ cm/sec.

Applying relation (2) on the first sphere

$$v_1' = v_1 + \frac{I}{m_1} = 500 + \frac{60000}{200} = 800 \text{ cm/sec.}$$

Applying relation (2) on the second sphere

$$v_2' = v_2 - \frac{I}{m_2} = 200 - \frac{60000}{200} = 600 \text{ cm/sec.}$$

Therefore the two spheres move after impact in the same direction as that before impact, the first with velocity 8 m/sec. and the second with velocity 6 m/sec.

Example (3) :

Two spheres are moving in the same straight line on a horizontal gorund, one of them towards the other. If the mass of the first is 100 gm and is moving with velocity 10 m/sec, the mass of the second is 300 gm and is moving with velocity 2 m/sec.

Find the velocity of the second sphere just after impact and its impulse on the first sphere given that the first sphere rebounds just after impact in a direction opposite to its original direction with a velocity 8 m/sec.

Solution :

Let the positive direction be the direction of motion of the first sphere, fig. (56)

$\therefore \quad v_1 = 10$ m/sec.

$v_2 = -2$ m/sec.

$v_1' = -8$ m/sec.

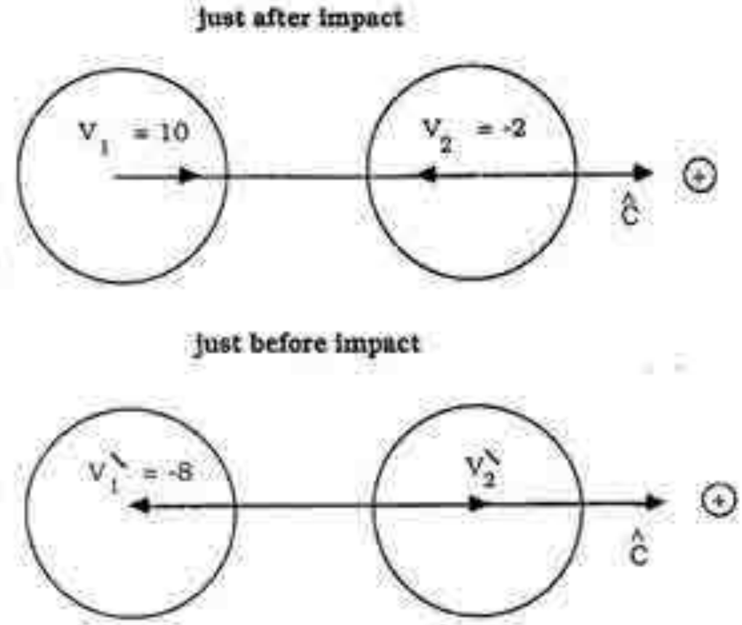


Fig. (56)

Expressing the masses in units of kilogram, we have $m_1 = 0.1$ kg, $m_2 = 0.3$ kg.

Applying relation (2) to the first sphere

$v_1' = v_1 + \frac{I}{m_1}$

$\therefore \quad -8 = 10 + \frac{I}{0.1}$

$\therefore I = -1.8$ newton.second

i.e. the magnitude of the impulse of the second sphere on the first sphere is equal to 1.8 newton, second.

Substituting in relation (2) for the second sphere

$v_2' = v_2 - \frac{I}{m_2}$

$= -2 - \frac{(-1.8)}{0.3} = 4$ m/sec.

which means that the second sphere rebounds after impact in a direction opposite to its original direciton with velocit 4 m/sec.

Exercises (2 - 3)

1. A railrood car of mass 21 tons moves with the velocity 14 m/sec. A barrier of collision brought it to rest through 0.3 second. Find the magnitude of the impulse and the magnitude of the average force in ton. wt.
2. A sphere of mass 50 gm falls from a height + of 2.5 metres on a horizontal ground and it rebounds from the ground to a height of 0.9 metre. Find average force between the sphere and the ground if the time during which they are in contact be 0.1 second.
3. A sphere of mass 500 gm falls from a height of 2.5 metres on a liquid surface and penetrated in it with a uniform velocily and described 3.5 metres in 2 minutes. Calculate the impulse of the liquid on the sphere.
4. A smooth sphere of mass 150 gm moves in a straight line on a horizontal ground with a velocity 0.6 m/sec. The sphere impinges with a vertical wall perpendicular to its direction of motion, it rebounds with a velocity 20 cm/sec, determine the impulse of the wall on the sphere.
5. A sphere of mass 400 gm moving on a horizonal ground with velocity 100 cm/sec. impinges directly with a vertical wall. If the wall acts on the sphere with an impulse of magnitude 0.48 newton. second, find the velocity of rebound of the sphere.
6. Two smooth spheres of masses 0.2 kg, 0.4 kg are moving on a horizontal ground, in the same straight line. If the velocity of the

first is 6 m/sec. and the velocity of the second is 8 m/sec. and in the same direction as the first. The two spheres collide, and the velocity of the first increased in magnitude by 2 m/sec. Find the velocity of the second just after impact and the magnitude of the impulse of each sphere on the other.

7. Two smooth spheres of masses 100 gm, 200 gm are moving on horizontal ground in the same straight line, the velocity of the first is 1 m/sec. and the velocity of the second is 2 m/sec. in an opposite direction. If the two spheres collide, so that the second sphere moves in the same direction with velocity 0.75 m/sec. after impact, find the velocity of the first sphere and the impulse of the second sphere on it.

8. A smooth sphere of mass 200 gm moves in a straight line on a horizontal table with velocity 60 cm/sec. The sphere impinges on a smooth sphere of mass 400 gm at rest on the table. If the first sphere came to rest as a result of impact, prove that the second moves with velocity 30 cm/sec. after impact, then find the magnitude of the mutual impulse between the two spheres.

9. Two bodies of masses 200 gm, 800 gm move in the same straight line on a horizontal table with a velocity 4 m/sec. in two opposite directions. If the two bodies move after impact as one body, find the velocity after impact.

10. Two smooth spheres of masses m, 2 m move on a smooth horizontal table in the same straight line and in the same direction, so that the smaller sphere is in front and moving with velocity 10 m/sec. and the bigger one at the back and moving with velocity 12 m/sec. After impact the smaller sphere moves in its previous direction of motion with velocity 12 m/sec, what is the velocity of the bigger one after impact ?

11. Two smooth spheres of equal mass are projected on a smooth horizontal table, such that they moves in the same straight line, the first with velocity 30 cm/sec. and the second with veloicty 20 cm/sec. in an oppositve direction to the first. If the second sphere rebounds after impact with velocity 10 cm/sec. Find the velocity of the first sphere after impact.

12. Two smooth spheres are projected on a smooth horizontal table so that they move in the same straight line and in the same direction. If the mass of the front sphere is 500 gm, and the magnitude of its velocity is 20 cm/sec, the mass of the back sphere is 200 gm and the magnitude of its velocity 50 cm/sec, find the velocity of the two spheres after impact given that they became one body.

13. Two smooth spheres each of mass 400 gm move in a straight line on a smooth horizontal table with velocity 4 m/sec. in the

same direction and there is a distance between them. A barrier is placed on the table so that it cuts the path of the two spheres at right angles, and the front sphere impinges with it and rebounds to collide with the back sphere, then it rebounds once more with velocity 2m/sec. Find the velocity of the back sphere after impact given that the barrier acts on the first sphere with an impulse of magnitude 2.8 newton second.

14. A ball of mass 1/2 kg falls from a height 3.6 m on a horizontal ground and rebounds to a height 1.6 m. Find the average force between the ball and the ground if the time during which they are in contact is 0.01 sec.

15. A bullet of mass 15 gm is fired with a velocity 1450.8 metres/minutes on a target at rest of mass 2 kg. If the bullet and the target moves as one body directly after the impact. Prove that the body move with a velocity 18 cm/sec after the impact.
If the body came to rest after having covered 81cm. find the resistance force given that it is constant.

16. A ball of mass 1/2 kg moving in a straight line with velocity 44 cm/sec. collides with another ball of mass $1\frac{1}{2}$ kg which is at rest. If they form one body after the impact, find its common velocity. if the body came to rest after having covered 11cm. find the resistance force given that it is constant.

17. A ball of mass 120 gm moving with uniform velocity 40 cm/sec. passes by a certain point, after one minute another body of mass 80 gm starting motion with uniform acceleration 4 cm/sec^2 and initial velocity 60 cm/sec in the same direction. If they form one body after the impact, find the common velocity, find also the time taken till this body comes to rest given that the resistance force is 3840 dyne .

18. A smooth inclined plane is of length AC = 19.6 metres and $B \in \overline{AC}$, AB = BC and the plane inclined at an angle of measure 30° to the horizontal. A ball of mass 3 gm is placed at A the top of the plane and moved on $\overline{AC}$ collides another ball of mass 1 gm at rest at B. If the two balls moved as one body after the impact, find the time taken after impact till the body reached C.

19. A ball of mass 1 kg. falled from a height 4.9m on a horizontal ground and rebounded to the maximum height 2.5m. Find the change in its momentum due to impact,then find the magnitude of the reaction of the ground on the ball if the time during which they are in contact is 0.1 sec.

Chapter Four

Work - Power - Energy

Preface :

In this chapter we will deal with: Work, Power and Energy .

Objectives:

By the end of teaching this chapter, the student should be able to:

(1) Recognize the work done by a force and its units.

(2) Recognize the concept of power, its units.

(3) Recognize the Kinetic energy and its units.

(4) Recognize the Principle of work and energy.

(5) Recognize the Potential energy.

Topics :

(1) Work.

(2) Power.

(3) Energy.

Work - Power - Energy

Work, power, energy are considered of the main concepts in mechanics, and closely related to our practical life. For example heat engines, motors consume energy and give us work which we use in different fields, and thus we have to determine precise definitions to these concepts, and try to extract the relations between these concepts and other motion characteristics such as displaceent velocity, acceleration and force.

In what follows, we are going to be concerned with the definiion of work, power, energy for a constant force, taking into consideration the possiblity of application on the motion under the action of the gravitational force, considered constant.

Notice that when we speak about a force acting on a particle we mean the resultant of the forces acting on it.

Work done by a constant force :

Consider a particle moving in a straight line under the action of a constant force $\vec{F}$, let the particle move from the initial position A to a new position B as in Fig. (57), and let its displacement vector be

$\vec{s} = \overrightarrow{AB}$

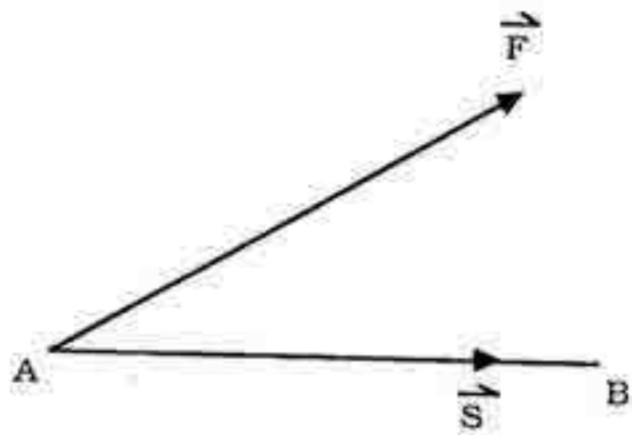


Fig. (57)

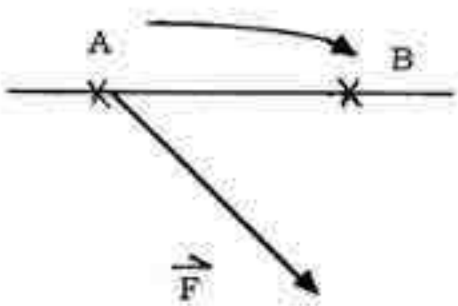
Definition :

The work done by the constant force $\vec{F}$ in moving the particle from an initial position to a final position, denoted by W is defined as the scalar product of the force vector times the displacement vector.

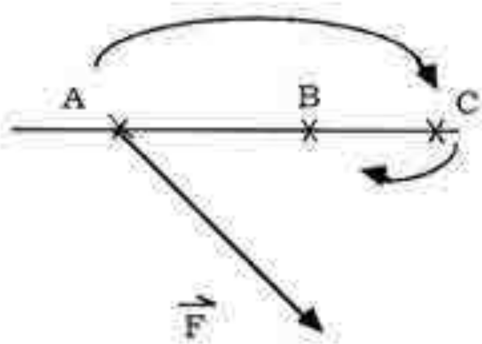
$\therefore \quad W = \vec{F} \odot \vec{a} \qquad (1)$

From this definition it is clear that the work is a scalar quantity, which may be positive, negative or zero, according to the magnitude and direction of both the two vectors $\vec{F}$, $\vec{s}$, and that work is defined between two positions (or betwen two time instants), one of them initial and the other final.

Also we can deduce directly from the definition that the magnitude of work is independent of the path on which the particle moves from the initial position to the final position, but depends only on these two positions. Fig. (58) shows two cases of motion of a particle, in which the work done is the same (provided the acting force is the same in both cases).



motion from A to B
Fig. (58-a)



motion from A to C . then from C to B
Fig. (58-b)

As a special case if a particle moves from a certain position and returns to the same position, the work done during the path is equal to zero.

This property can be easily verified if we notice that the displacement vector in this case in the zero vector.

If θ is the measure of angle between the two vectors $\overrightarrow{F}$, $\overrightarrow{s}$ drawn from the same point Fig. (59), relation (1) can be written in the form :

$$W = \|\overrightarrow{F}\| \; \|\overrightarrow{S}\| \cos\theta \quad (2)$$

Fig. (60) shows some different cases of the two vectors of force and displacement.

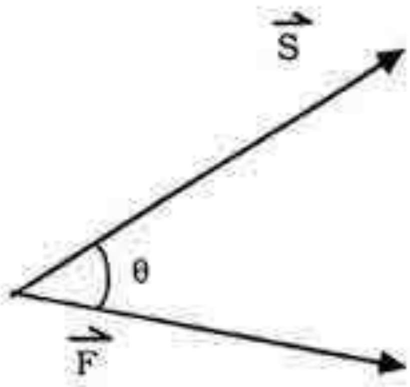


Fig. (59)

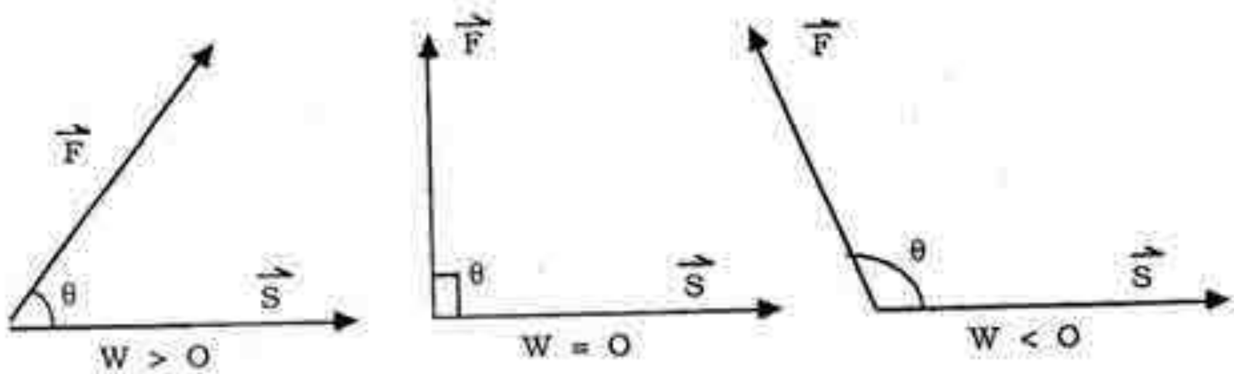


Fig. (60)

N.B.:

When the work done between two positions is negative we call it a resisting work, as an example, the work done by a resisting force on the body or force of friction, which will be illustrated through the examples.

Example (1) :

a particle moves in a straight line under the action of the force $\overrightarrow{F} = 5\,\hat{i} - 3\,\hat{j}$, from the point A = (1 , 0) to the point B = (3 , 3) where coordinates are referred to a system of rectangular cartesian coordinates $\overline{OX}$, $\overline{OY}$. Find the work done.

Solution :

Fig. (61) shows the position of the two points A, B lreferred to the axes.

To calculate the displacement vector we notice that $\overrightarrow{s} = \overrightarrow{AO} + \overrightarrow{OB} = \overrightarrow{OB} - \overrightarrow{OA}$

$\therefore \quad \overrightarrow{s} = (3 - 1)\,\hat{i} + (3 - 0)\,\hat{j}$

$\quad = 2\,\hat{i} + 3\,\hat{j}$

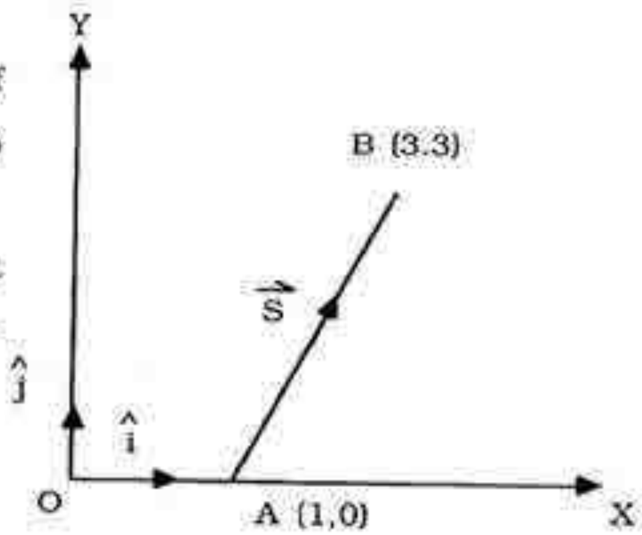


Fig. (61)

Applying the definition of work, noticing that the given force is constant

$$\therefore \quad W = \overrightarrow{F} \odot \overrightarrow{s}$$

$$= (5\,\hat{i} - 3\,\hat{j}) \odot (2\,\hat{i} + 3\,\hat{j})$$

$$= 5 \times 2 + (-3) \times (3) = 1 \text{ unit of work.}$$

Rule :

If two subsequent displacements occur to a body under the action of a force, then the work done during the resultant displacement is equal to the sum of the work done during each of the two displacements.

Proof :

Let W_1 , W_2 be the work done during the two displacements $\overrightarrow{s_1}$, $\overrightarrow{s_2}$ respectively, W the work done during the resultant displacement Fig. (62)

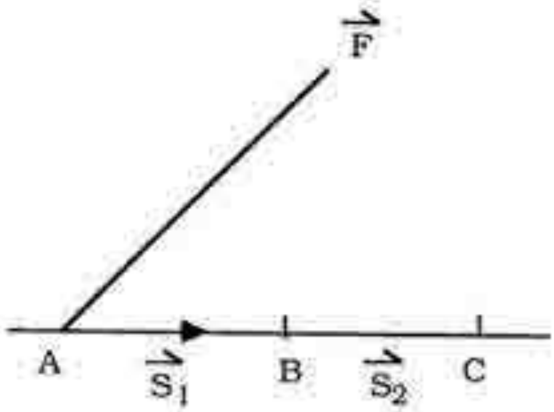


Fig. (62)

From the definition

$$W_1 = \overrightarrow{F} \odot \overrightarrow{s_1} \ , W_2 = \overrightarrow{F} \odot \overrightarrow{s_2}$$

If the body moves from A to B during the first displacement and from B to C during the following displacement, then the resultant displacement means the motion from A to C which corresponds to a displacement $(\overrightarrow{s_1} + \overrightarrow{s_2})$.

$$\therefore \quad W = \overrightarrow{F} \odot (\overrightarrow{s_1} + \overrightarrow{s_2}) = \overrightarrow{F} \odot \overrightarrow{s_1} + \overrightarrow{F} \odot \overrightarrow{s_2}$$

$$= W_1 + W_2 \qquad \text{Q.E.D.}$$

We notice that the rule is valid also for any finite number of consecutive displacements.

Example (2) :

A particle moves in a straight line from position A to a new position B, then it returns to its original position. Apply the previous rule to show that the total work done is equal to zero.

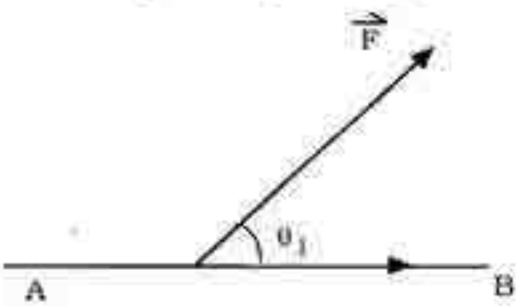
Solution :

Let W_1 , W_2 be the work done during the displacement from A to B, and from B to A respectively by the force $\overrightarrow{F}$

$W_1 = \overrightarrow{F} \odot \overrightarrow{AB}$ $\qquad W_2 = \overrightarrow{F} \odot \overrightarrow{BA}$

$$\begin{aligned} W_1 + W_2 &= \overrightarrow{F} \odot \overrightarrow{AB} + \overrightarrow{F} \odot \overrightarrow{BA} \\ &= \overrightarrow{F} \odot (\overrightarrow{AB} + \overrightarrow{BA}) \\ &= \overrightarrow{F} \odot \overrightarrow{0} = 0 \end{aligned}$$

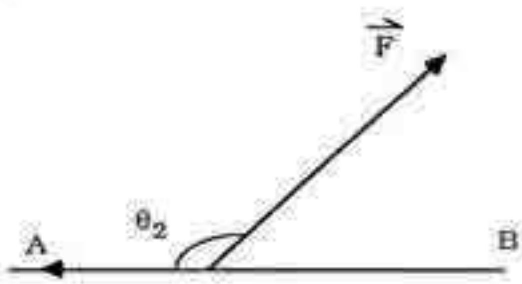
Figure (63) shows how to calculate the two works W_1 , W_2 using relation (2) for calculating the scalar product to the two vectors $\overrightarrow{F}$, $\overrightarrow{s}$. θ_1 , θ_2 denote the measures of the angles betwen $\overrightarrow{F}$ and $\overrightarrow{AB}$ and $\overrightarrow{BA}$ respectively during the two displacements.



Displacement from A to B

$W_1 = ||\overrightarrow{F}|| \times AB \times \cos\theta_1$

Fig. (63-a)



Displacement from B to A

$W_2 = ||\overrightarrow{F}|| \times AB \times \cos\theta_2$

$= ||F|| \times AB \times \cos(180-\theta)$

$= -W_1$

Fig. (63-b)

If the force is parallel to the displacement, we can express each of these two vectors in terms of its algebraic measure relative to a unit vector $\hat{c}$ as follows.

$\vec{F} = F\,\hat{c} \quad , \quad \vec{s} = s\,\hat{c}$

Thus the work is calculated as follows

$$W = \vec{F} \odot \vec{s} = (F\,\hat{c}) \odot (s\,\hat{c})$$

$$= F\,s\,(\hat{c} \odot \hat{c})$$

$$\because \quad (\hat{c} \odot \hat{c}) = 1$$

$$\therefore \quad W = F\,s \qquad (3)$$

This relation expresses that "if the force vector is parallel to the displacement vector then the work done is equal the product of their algebraic measures".

Unit of measure of work :

From the definition we deduce directly that

Unit of measure of work

= unit of measure of magnitude of force × unit of length

The newton. metre (newton . metre) :

The newton. metre is defined as the magnitude of work done by a force of magnitude one newton in moving a body a distance one metre in its direction.

$\therefore$ If we take $F = 1$ newton

$s = 1$ metre

$\theta = 0$ in relation (2)

We get

1 newton metre = 1 newton × 1 metre

The unit of newton . metre is called the joule.

Kilogram-metre. (Kg.wt.m.) :

The kilogram. metre is defined as the magnitude of the work done by a force of magnitude one kilogram weight in moving a body a distance one metre in its direction.

If we take $F = 1$ kg. wt, $s = 1$ m,

$\theta = 0$ in relation (2)

We get

1 kg.wt.m. = 1 kg.wt. × 1 metre.

Erg.

The erg is defined as the magnitude of the work done by a force of magnitude one dyne in moving a body a distance one centimetre in its direction.

If we take $F = 1$ dyne.

$s = 1$ cm.

$\theta = 0$ in relation (2), we get

1 erg = 1 dyne × 1 centimetre

Here are the rules for transferring between different units of work :

1 kg.wt.m. = 1 kg.wt. × 1 metre = 9.8 newton × 1 metre

= 9.8 newton . metre (or joule).

1 newton . metre or joule = 1 newton × 1 metre

= 10^5 dyne × 10^2 cm. = 10^7 dyne. cm.

= 10^7 ergs.

Example (3) :

A man whose mass is 80 kg ascends an inclined plane of inclination 30° and of length 100m. Find the work done by his weight.

Solution :

Applying relation (2). noticing that

F = m g = 80 kg.wt.

s = 100 m, θ = 120°

Fig. (64)

W = 80 × 100 × cos 120°

= 80 × 100 × (-1/2)

= - 4000 kg.wt. m.

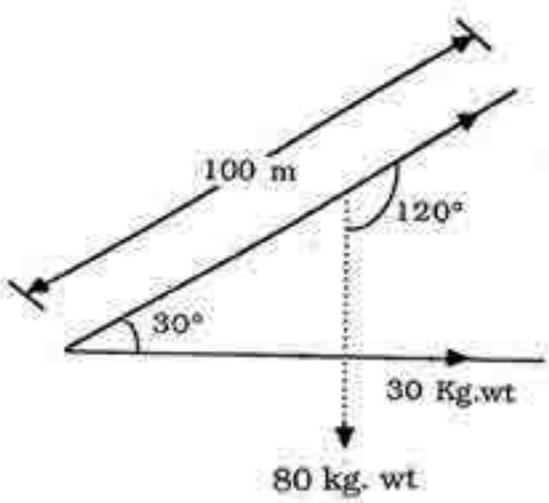


Fig. (64)

Example (4) :

A particle moves in a straight line. A resistance force of magnitude 100 newtons acts on it. Find the work done by this force during a displacement of magnitude 300 metres.

Solution :

Since the force is a force of resistance, then it acts in an opposite direction to the displacement vector, and if $\hat{c}$ is a unit vector in the direction of the displacement, we can express each of the displacement and force vectors in terms of their algebraic measures.

Fig. (65)

$\overrightarrow{s} = s\,\hat{c}$, $\overrightarrow{F} = F\,\hat{c}$

In our case $s = 300$ m, $F = -100$ newtons.

Fig. (65).

$\therefore\ W = F \times s$

$= (-100) \times (300) = -3 \times 10^4$ newton metre.

$= -3 \times 10^4$ joules.

Example (5) :

A particle of mass m moves on the line of greatest slope of a plane whose inclination to the horizontal is θ. Find the work done by the weight when the particle is ;

1. ascending the plance
2. descending the plane

Solution :

Let g be the magnitude of the gravitational acceleration. The weight, is a force of magnitude mg acting vertically downwards. Resolve this force in two perpendicular directions, one of them is parallel to the line of greatest slope, and the second perpendicular to it and in the vertical plane passing through the line of greatest slope, as in Fig. (66).

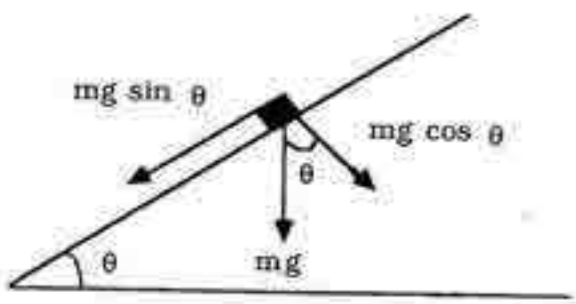


Fig. (66)

The component of the weight force in the first direction is of magnitude (m g sin θ) parallel to a line of greatest slope downwards.

The component of tne weight force in the second direction is of magnitude ($m\ g\ \sin\theta$) perpendicular to a line of greatest slope and to the inside of the plane.

If $\overrightarrow{F}$ is the weight force, $\overrightarrow{F_1}$, $\overrightarrow{F_2}$ the two components mentioned above, then

$$\overrightarrow{F} = \overrightarrow{F_1} + \overrightarrow{F_2}$$

Consider the motion of the particle along a line of greatest slope, and let $\overrightarrow{s}$ be its vector displacement. The work done by the weight is:

$$W = \overrightarrow{F} \odot \overrightarrow{s} = (\overrightarrow{F_1} + \overrightarrow{F_2}) \odot \overrightarrow{s}$$
$$= \overrightarrow{F_1} \odot \overrightarrow{s} + \overrightarrow{F_2} \odot \overrightarrow{s}$$

but $\overrightarrow{F_2} \cdot \overrightarrow{s} = 0$ since the component $\overrightarrow{F_2}$ is perpendicular to the displacement vector.

$$\therefore \quad W = \overrightarrow{F_1} \cdot \overrightarrow{s}$$

i.e. the work done by the weight force is equal to the work done by the component parallel to the line of greatest slope.

Now consider the two cases of ascending and descending.

1. If the particle is ascending:

Consider a unit vector $\hat{c}$ parallel to a line of greatest slope, and directed upwards as in Fig. (67). We can express each of the vectors $\overrightarrow{s}$, $\overrightarrow{F_1}$ in terms of its algebraic measure as follows :

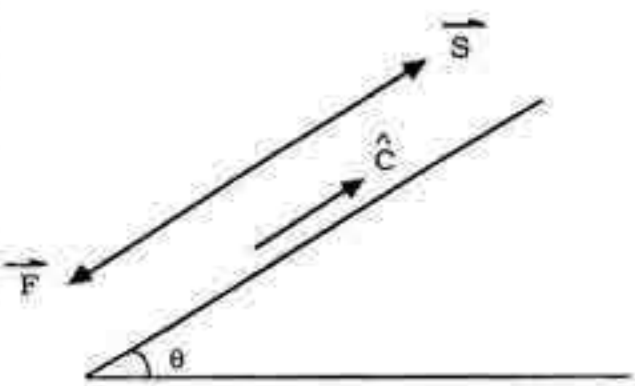


Fig. (67)

$\overrightarrow{s} = L\ \hat{c}$, $\overrightarrow{F_1} = -(m\ g\sin\theta)\ \hat{c}$

$$W = L\,\hat{c} \odot (-m\,g\sin\theta)\,\hat{c}$$
$$= -\,m\,g\,L\sin\theta\,(\hat{c}\odot\hat{c})$$
$$= -\,mgL\sin\theta.$$

2. If the particle is descending :

Consider a unit vector $\hat{c}$ parallel to a line of greatest slope and directed downwards, as in Fig. (68).

In this case :

$$\vec{s} = L\,\hat{c}\ ,\ \overrightarrow{F_1} = (m\,g\sin\theta)\,\hat{c}$$
$$\therefore\ W = L\,\hat{c}\odot(m\,g\sin\theta)\,\hat{c}$$
$$= m\,g\,L\sin\theta\,(\hat{c}\odot\hat{c})$$
$$= m\,g\,L\sin\theta$$

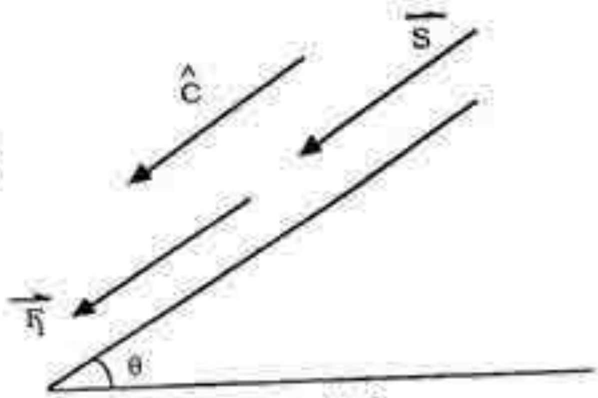


Fig. (68)

Example (6) :

A body of mass 2kg. is projected with velocity 1.4m/sec. Upwards along the line of greatest slope of a smooth inclined plane whose inclination to the horizontal is θ°where $\sin\theta = \frac{1}{98}$. Find the work done by the body until it comes to rest.

Solution :

The body moves upwards by a uniform retardation = g sin θ

i.e. $a = -\,9.8 \times \frac{1}{98} = -\,0.1\ m/sec^2$

$\because\ v^2 = v_0^2 + 2\,a\,s$

$\therefore\ o = (1.4)^2 - 2 \times 0.1 \times s$

$\therefore\ s = \frac{(1.4)^2}{0.2}$

$= 9.8\ m$

The work = $-\,mg\sin\theta \times s$

$= -2 \times 9.8 \times \frac{1}{98} \times 9.8$

$= -\,1.96$ newtons .m

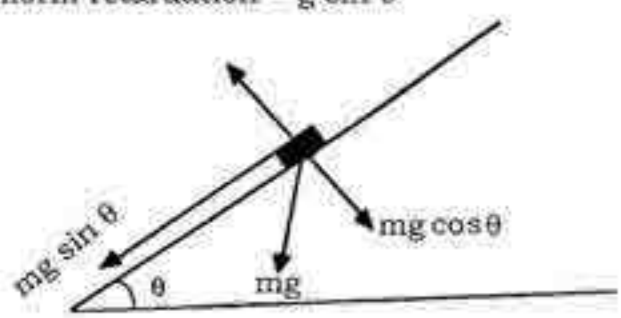


Fig. (69)

Exercises (4 - 1)

1. A particle moves in a straight line from the point $A = (-1, -2)$ to the point $B(1, -3)$ under the action of a force $\overrightarrow{F} = -\hat{i} - \frac{1}{2}\hat{j}$, the resolution being referred to two perpendicular directions $\overrightarrow{OX}$, $\overrightarrow{OX}$ and $\hat{i}$, $\hat{j}$ are unit vectors in these two directions.

   Calculate the work done by this force.

2. A particle moves from the origin $O = (0, 0)$ to the point $A = (2, 0)$ along a straight line, then to the point $B = (7, 2)$ on a straight line also under the action of a force $\overrightarrow{F} = -4\hat{i}$. Calculate the work done by this force during each of these two displacements, then prove that the sum of the two works is equal to the work done along the resultant displacement.

3. A tram car is pulled by a horizontal rope inclined at an angle of $60°$ to the tram's track, and it moved a distance of 15m. If the tension in the rope is equal to 150 kg.wt, find the work done by the tension force measured in ergs.

4. A man ascends a straight road inclined at an angle of $30°$ to the horizontal, he moved a distance 300m, then returned to the starting point. Find the work done by its weight throughout the whole journey. If the resistance force to the motion of the man is equal to 2kg.wt during his motion, find the work done by this force during the whole journey.

5. A force $\overrightarrow{F} = 2\hat{i} + 3\hat{j}$ acts on a particle so that its position vector

$\overrightarrow{R} = (t + 5)\ \hat{i} + (t^2 + 4)\ \hat{j}$ where $\hat{i}, \hat{j}$ are two perpendicular unit vectors, F measured in newtons, distance in cm. Calculate the work done by this force between t = 1, t = 5.

6. A particle moves upwards along a line of greatest : slope of a plane inclined at an angle 30° to the horizontal a distance 400cm under the action of a force of magnitude 40 newtons in the vertical plane passing through the line of greatest slope and inclined to the horizontal at an angle 60° upwards. Find the work done.

7. A body of mass 2 kg is placed on a smooth inclined plane whose inclination to the horizontal is 30°. Find the work done by its weight when the body moves a distance 5 m along the line of greatest slope, measured in joules.

8. A particle moves in a straight line under the action of force of magnitude 400 dynes acting in the direction of motion. Find the work done by this force during a displacement of magnitude 300 cm.

9. Find thw work done by the weight when a body of mass 4 tons is raised up words a distance of 12 metres.

10. Find the work done when a mass of magnitude 100 gm moves a distance of 150 cm with an accelration of magnitude 5 cm/sec^2.

11. A car of mass 6 tons ascends a slope making an angle of sin $\frac{1}{100}$, with the horizontal, the resistance is 10 kg.wt per ton of mass of the car, its velocity amounted to 63 km./h after $12\frac{1}{2}$ seconds. Find the work done in this interval:

    i. by the engine force.

    ii. by resistances.

iii. by the weight of the car.

iv. against the weight of the car.

12. A body of mass 100 gm is placed at the top of a rough inclined plane whose height is one metre. If the body slides down from rest and reached the bottom of the plane with velocity 180 m/sec, find the work done against friction.

13. A body of mass 14kg moves from rest on a horizontal road under the effect of a force $\overrightarrow{F}$ of magnitude 0.95 kg.wt and inclined to the horizontal by an angle 60
resistance of magnitude 0.95 kg.wt, find in joules the work done through the first minute by each of the following forces :
1) The weight of the body 2) $\overrightarrow{F}$ 3) The resistance

Power

Usually when a force do work along a certain path, i.e. during a certain interval of time, this work is not done regularly, which means that it is not necessary that the force does equal amounts of work during equal intervals of time. Therfore it is very important to ask about the "time rate of doing work by this force".

Defintion

"Power is the time rate of doing work".

This defintion can be fromulated also in the following :

"Power is the work done in unit time".

Since the time rate of doing work is given by the derivative the work function with respect to the time, therfore:

$$\text{Power} = \frac{d\,W}{d\,t} \qquad (1)$$

In what follows we will be concerned with the study of the motion of a particle in a straight line, and we will assume that the

Thus it is possible to express the displacement, velocity and force vectors in terms of their algebraci measures relative to a unit vector $\hat{c}$ parallel to the straight line on which motion occurs :

Let $\overrightarrow{s} = s\,\hat{c}$, $\overrightarrow{v} = v\,\hat{c}$, $\overrightarrow{F} = F\,\hat{c}$

$$\therefore\ W = F\,s$$

$$\therefore\ \text{Power} = \frac{d}{d\,t}\,(Fs)$$

but F is constant (since the force vector is constant by assumption) and thus can be taken out, therefore.

$$\text{Power} = F\,\frac{d\,s}{d\,t}$$

Since $\frac{d\,s}{d\,t} = v$ = algebraic measure to velocity vector

$$\therefore\ \text{Power} = F\,v \qquad (2)$$

i.e. "Power is equal to the product of the algebraic measures of force and velocity vectors".

Relation (2) shows that power is a scalar quantity to be determined at any instant if given the velocity and force vectors (or their algebraic measures) at this instant, and its value gives the time rate of doing work.

N.B.

The student has to notice that power is determined at an instant, while work is always calculated between two instants.

Units of measure of power :

Since power is the time rate of doing work, therefore

$$\text{Unit of measure of power} = \frac{\text{unit of measure of work}}{\text{unit of measure of time}}$$

We can also determine the unit of measure of power according to relation (2) as follows :

Unit of measure of power = unit of measure of force

$\times$ unit of measure of velocity

Newton. metre / second (newton. m/sec)

Newton. metre / second is defined as the power of a force doing work at a constant time rate of magnitude newton metre every second.

newton. metre / second is called the watt.

Kilogram. metre / second (kg. wt.m/sec.

Kilogram. metre / second is defined as the power of a force doing work at a constant time rate of magnitude one kilogram. metre every second.

Erg/second (erg/sec) : It is defined as the power of a force doing work at a constant time rate of magnitude one erg every second.

Here are the rules for transferring between different units of power.

1 kg. wt. m/sec = 9.8 newton. m/sec.

1 newton. m/sec = 1 watt = 10^7 erg/sec.

There are also other units of power such as kilowatt. hourse

1 kilowatt = 1000 watt = 1000 newton. metre / second

= 10^{10} erg/sec.

1 hourse = 75 kg. wt.m/sec. = 75 $\times$ 9.8 newton. m/sec.

= 735 newton. metre / second (or watt.) = 0.735 kilowatt.

Example (1) :

A train of mass 200 tons and the power of its engine is 400 horses moves in a straight line on a horizontal ground with maximum velocity of magnitude 54 km/h. Find the magnitude of the road resistance to its motion and the magnitude of resistance per ton of its mass.

Solution :

Chose a unit vector $\hat{c}$ in the direction of motion of the train. i.e. in direction of force generated by the motor. Twe forces act horizontally on the train :

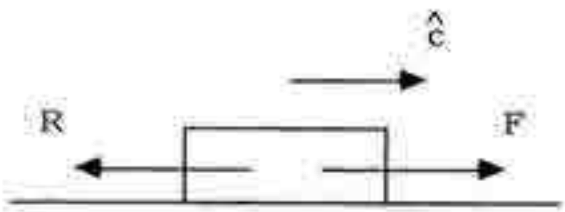


Fig. (70)

i- The force generated by the motor and its direction is the direction of motion. i.e. in drection of $\hat{c}$, let F be the magnitude of this force which is also its algebraic measure relative to the unit vector $\hat{c}$.

ii- The force of resistance to the motion of the train which is due to the contact between the wheels of the train and the rails, air resistance, which we call "road resistance to motion" and this force is directed opposite to the direction of motion, let R be its magnitude.

Since the train is moving uniformly (magnitude of velocity constant and equal to maximum velocity).

$$\therefore R = F$$

We can determine the value of F since we are given the power as follows :

First we express the power in units of kg. wt.m/sec.

Power = 400 horses = 400 × 75 = 30000 kg. wt. m/sec.

The magnitude of the velocity vector (which is equal to its algebraic measure at the same time) is

$$v = 54 \text{ km/h}$$

$$= 54 \times \frac{5}{18} = 15 \text{ m/sec.}$$

$$\text{Power} = F\,v$$

$$\therefore \quad F = \frac{\text{Power}}{v} = \frac{30000}{15} = 2000 \text{ kg.wt}$$

$\therefore$ Magnitude of resistance = R = F = 2000 kg.wt.

Magnitude of resistance per ton of mass of train

$$= \frac{\text{magnitude of resistance}}{\text{mass of train}}$$

$$= \frac{2000}{200} = 10 \text{ kg. wt.}$$

Example (2) :

A car of mass 1710 kg, the power of its motor is 12 horses, is moving along a horizontal straight road with its maximum velocity of magnitude 72 kg/h. what is the maximum velocity with which this car could ascend a straight road inclined at an angle of mesure α to the horizontal so that $\sin \alpha = \frac{1}{10}$ given that the resistance is the same on the two roads ?

Solution :

1. Motion on the horizontal road :

Let F be the magnitude of the force generated by the motor of the car on the horizontal road. R the magnitude of the resistance force of the road to the motion.

Since the motion is uniform $\therefore$ R = F

Power = 12 horses = 12 × 75 = 900 kg. wt. m/sec.

$V = 72$ km/h. $= 72 \times \frac{5}{18} = 20$ m/sec.

$F = \frac{\text{Power}}{v} = \frac{900}{20} = 45$ kg. wt.

2. Motion on the inclined road :

Let α be the angle of inclinao the horizontal. Choose a unit vector $\hat{c}$ in the direction of motion as in fig. (71).

Let F_1 be the magnitude of the force generated by the motor of the car on the inclined road

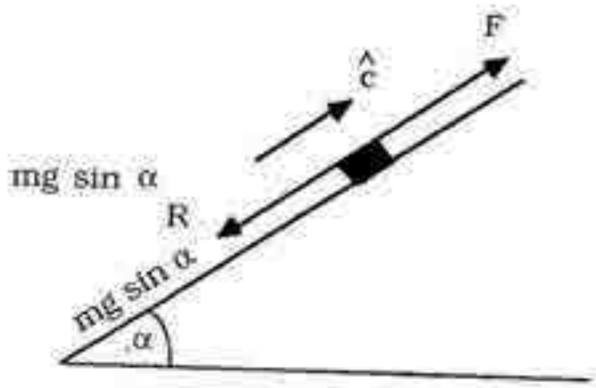


Fig. (71)

(which is equal to its algebraic measure at the same time). R_1 equal to the algebraic measure of the velocity vector.

The resistance is composed of

i- resistance force of the road and is equal to 45 kg. wt.

ii- component of the weight force acting in a direction opposite to the direction of motion and of magnitude.

$$m\,g\sin\alpha = 1710 \times \frac{1}{10} = 171 \text{ kg. wt.}$$

$$\therefore \quad R_1 = 45 + 171 = 216 \text{ kg. wt.}$$

Since the motion is uniform

$$\therefore \quad F_1 = R_1 = 216 \text{ kg. wt.}$$

But the power of the motor is the same as in the case of motion on the horizontal road.

$$\therefore \quad \text{power} = F_1 v_1 = 900 \text{ kg. wt. m/sec.}$$

$$\therefore \quad v_1 = \frac{\text{power}}{F_1} = \frac{900}{216} = \frac{25}{6} \text{ m/sec.}$$

$$= \frac{25}{6} \times \frac{18}{5} = 15 \text{ km/h.}$$

Example (3) :

An aeroplane flies in a horizontal path under the action of a resistance proportional to the square of its speed. If the magnitude of the resistance is 600 kg. wt when its velocity is 200 km/h and the maximum speed of the aeroplane is 300 km/h, find the power of its engine measured in horses.

Solution :

Let R_1 , R_2 be the magnitudes of the resistance forces when the speeds are 200 , 300 km/h respectively k is the constant of proportionality between the magnitude of the resistance and the square of the speed.

$$\therefore \quad R_1 = k\,(200)^2$$

$$R_2 = k\,(300)^2$$

$$\therefore \quad \frac{R_2}{R_1} = \frac{(300)^2}{(200)^2} = \frac{9}{4}$$

but $R_1 = 600$ kg. $\therefore$ $R_2 = 9 \times 600 = 1350$ kg. wt.

R_2 is the magnitude of the force generated by the engine at maximum speed.

Maximum speed $= V = 300 \times \frac{5}{18} = \frac{250}{3}$ m/sec.

Power $= F \times v = 1350 \times \frac{250}{3}$ kg. wt. m/sec.

$$= \frac{1350 \times 250}{3 \times 75} = 1500 \text{ horses}$$

Example (4) :

A car of mass 6 tons is moving with maximum velocity of magnitude 27 km/h up an inclined road whose inclination to the horizontal is an angle whose sine is $\frac{1}{100}$. The car then returned with its maximum velocity of magnitude 135 km/h. Find the force of resistance of the road if it does not change all time, then find the power of the motor.

Solution :

First : In the ease when the car is ascending :

Let $\hat{c}$ be a unit vector in the direction of motion, α the angle of inclination of the road to the horizontal. F_1 is the magnitude of the force generated by the motor, when the car is moving up the plane and is equal to the algebraic measure of this force at the same time.

R the magnitude of the road resistance to motion, as in fig. (72)

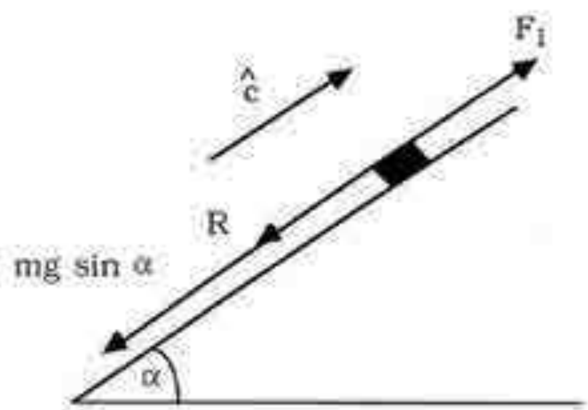


Fig. (72)

The total resistance force is composed of the resistance of the road and the component of weight force, which is in a direction opposite to the direction of motion and of magnitude $m\ g\ \sin\alpha$ = $6000\ g \times \frac{1}{100} = 60$ kg. wt.

Since the motion is uniform

$\therefore\ F_1 = R + 60$

The magnitude of the velocity (which is equal to the algebraic measure of the veloity vector) is

$v_1 = 27 \times 5$ m/sec.

The power of the motor $= F_1\ v_1 = (R + 60) \times 27 \times \frac{5}{18}$

kg. wt. m/sec.

Second : In the case when the car is descending :

Let $\hat{c}$ be a unit vector in the direction of motion, let F_2 be the magnitude of the force generated by the motor during the motion downwards (which is equal to the algebraic measure of this force) as in fig. (73).

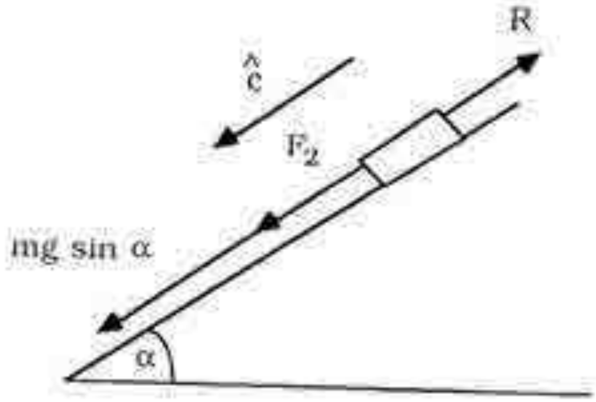


Fig. (73)

Notice that the component of the weight parallel to the plane will be in the direction of motion, i.e. it is in the same direction as the force generated by the motor. Since the motion is uniform.

$\therefore\ F_2 + 60 = R \qquad \therefore\ F_2 = R - 60$

The magnitude of the velocity (which is equal to the algebraic measure of the velocity vector) is

$v_2 = 135$ km/h $= 135 \times \frac{5}{18}$ m/sec.

The power of the motor $= F_2\ v_2 = (R-60) \times 135 \times \frac{5}{18}$ kg. wt. m/sec.

Equating the two values of power obtained in cases of ascending and descending.

$(R + 60) \times 27 \times \frac{5}{18} = (R - 60) \times 135 \times \frac{5}{18}$

$R + 60 = 5\ (R - 60)$

$\therefore \qquad R = 90$ kg. wt.

Substituting in the relation giving the power, we get

$$\begin{aligned} \text{power} &= (R - 60) \times 135 \times \frac{5}{18} \\ &= (90 - 60) \times 135 \times \frac{5}{18} \\ &= 1125 \text{ kg. wt. m/sec.} \\ &= \frac{1125}{75} = 15 \text{ horses.} \end{aligned}$$

Example (5) :

A car of mass 1 ton moves by a uniform velocity of magnitude 54 km/h on a horizontal road. If the same car moves up with the same velocity on inclined plane which makes with horizontal an angle whose sine equals $\frac{1}{50}$, find the increase in the power engine in horses, given that the magnitude of the resistance is the same on the road and the plane.

Solution :

1st The motion on the road :

$F = R$ where R is the resistance.

The power $= F \times v$

$= R \times 54 \times \frac{5}{18} = 15\ R$ kg.wt.m/sec.

2^{nd} The motion on the inclind plane :

$F = R + 100 \times \frac{1}{50} = R + 20$ kg.wt.

$\therefore$ The power = $(R + 20) \times 15$ kg.wt. m/sec.

$\therefore$ The increase in the power $= 15\,R + 300 - 15\,R$

$= 300$ kg.wt. m/sec.

$= 4$ horses.

Exercises (4 - 2)

1. If the displacement vector as a function of the time of a particle of unit mass is $\vec{s} = t\,\hat{i} + (\frac{5}{2}t^2 - 3\,t)\,\hat{j}$, where $\hat{i}$, $\hat{j}$ are tow perpendicular unit vectors in the two directions $\overrightarrow{OX}$, $\overrightarrow{OY}$ respectively, if the force acting on this particle is $\vec{F} = 3\,\hat{i} + 4\,\hat{j}$ find $\frac{d}{d\,t}(\vec{F} \odot \vec{S})$ at $t = 3$.
2. Find in kilowatts and in horses the power of a car of mass 2 tons when it moves with velocity 50 kg/h on a horizontal road, given that the force of resistance is equivalent to 0.05 of the weight of the car.
3. A tram of mass 250 tons is moving on horizontal rails with uniform velocity 30 km/h. Find the power of the train's engine, given that the road resistance is equal to 9 kg. wt per ton of its mass.
4. The power of the engine of a train of mass 200 tons is 300 horses It the train is moving on horizontal rails, find the maximum velocity of the train given that the magnitude of the resistance is equal to 0.015 of the train's weight.
5. A lorry of mass 4 tons and the power of whose engine is 20 horses is moving up an inclined road whose inclination to the horizontal an angle of measure θ, where $\sin\theta = \frac{1}{20}$. What is the maximum velocity of the lorry on the road, given that the

magnitude of the road resistance to the motion is 25 kg. wt for each ton to the mass of the lorry ?

6. A train of mass 200 tons is moving along a horizontal road with its maximum velocity of magnitude 90 km/h, the resistance force to its motioin being 10 kg. wt for each ton of its mass. If the train begins to move up a road inclined to the horizontal at an angle of measure θ, where $\sin \theta = \frac{1}{10}$, find the maximum velocity of the train on the inclined road, given that the resistance does not change.

7. If the maximum velocity of a bicycle on a horizontal road is 24 km/h, what is the resistance it suffers, given that the power of the bicycle's rider is $\frac{1}{5}$ horse, and if the mass of the bicycle and the rider is 72 kg, what is the maximum velocity with which the bicycle would go up a road inclined to the horizontal at an angle whose sine is $\frac{1}{16}$, given that the road resistance does not change.

8. A locomotive whose power is 400 horses pulls a train with velocity 72 km/h on a horizontal ground. Find the magnitude of the resistance per ton of mass of train, If the mass of the train and locomotive together is 200 tons, find the maximum velocity with which the train would go up a road inclined to the horizontal at an angle whose sine is $\frac{1}{200}$, assuming that the road resistance does not change.

9. The power of the engines of an aeroplane is 600 horses. The aeroplane flies under the action of a resistance proportional to the square of its speed. If the maximum velocity of the aeroplane of 300 km/h, what is the magnitude of the resistance when its velocity is 200 km/h ?

10. A lorry of mass 2 tons descends an inclined road whose inclination to the horizontal is an angle whose sine is $\frac{1}{10}$ from position A to position B with a maximum velocity of 45 km/h. Find the power of the motor of the lorry given that the road resistance to its motion is equal 13% of the weight of the lorry. When the car reaches position B it is loaded with a mass of $\frac{1}{2}$ ton, then it moved upwards to position A with maximum velocity. Find this velocity if the ratio of the resistance to the weight remains constant.

11. A small aeroplane works at the rate of 25000 kg. wt. m/sec when its velocity is 90 km/h. If the resistance of air to the aeroplane is proportional to the square of its velocity, find the power exerted when the aeroplane moves with a velocity of 135 km/h in the same case.

12. A car of mass 6 tons is moving up a road inclined to the horizontal at an angle of measure θ, where $\sin \theta = \frac{1}{100}$, the resistance force to its motion being 15 kg. wt for each ton of its mass. If the

maximum velocity of the car is 27 km/h. calculate the horse power of the engine and also maximum velocity with which the car would go down the inclined road assuming that the power and the road resistance do not change.

13. A car of mass 27 tons moves on a horizontal plane with maximum velocity 100 km/h, and when it reached a road whose inclination to the horizontal is an angle whose sine $\frac{1}{20}$ it began to move down the road with the same velocity. If the resistance on the two roads is the same, find the power engine in horses.

KINETIC ENERGY

Definition :

The kinetic energy of particle, denoted by T is defined as the product of half the mass of the particle times the square of the magnitude of its velocity.

M is the mass of the particle, $\vec{v}$ is its velocity vector. v is the algebraic measure of this vector, then.

$$\boxed{T = \frac{1}{2} m \|\vec{v}\|^2 = \frac{1}{2} m v^2} \qquad (1)$$

$\|\vec{v}\|^2 = \vec{v} \odot \vec{v}$ we can express the kinetic energy as follow :

$$T = \frac{1}{2} m (\vec{v} \odot \vec{v})$$

From the above definitio it is clear that the kinetic energy of a particle is a positive scalar quantity, which vanishes only when the velocity vector vanishes. Also this definition shows that the kinetic energy of a particle may change from instant to instant during its motioin according to the magnitude of its velocity.

Units of measure of kinetic energy.

From the definition we deduce that :

Unit of measure of kinetic enery =

unit of measure of mass × (unit of measure of magnitude of velocity)^2.

or in other from :

Unit of measure of kinetic energy

$$= \text{unit of measure of mass} \times \frac{\text{unit of measure of length}}{\text{unit of measure of time}} \times \frac{\text{unit of measure of length}}{\text{unit of measure of time}}$$

$$= \text{unit of measure of mass} \times \frac{\text{unit of measure of length}}{(\text{unit of measure of time})^2} \times \text{unit of measure of length}$$

$$= \text{unit of measure of mass} \times \text{unit of measure of acceleration} \times \text{unit of measure of length}$$

$$= \text{unit of measure of magnitude of force} \times \text{unit of measure of length}$$

$$= \text{unit of measure of work}$$

$\therefore$ unit of measure of kinetic energy = unit of measure of work.

This shows that kinetic energy has the same nature of work.

As an example. If the mass is measured in kilograms and magnitude of velocity in m/sec then.

Unit of measure of kinetic energy

$$= \text{kg} \times \frac{\text{m}}{\text{sec}} \times \frac{\text{m}}{\text{sec}} = \text{kg} \times \frac{\text{m}}{\text{sec}^2} \times \text{m.} = \text{newton. m.}$$

If the mass is measured in gm and magnitude of velocity in cm/sec, then.

$$mg \times \frac{cm}{sec} \times \frac{cm}{sec} = gm. \times \frac{m}{sec^2} \times cm.$$
$$= dyne \times cm = erg$$

1. "metre - kilogram-second" system (m.k.s.)

In this system lengths are measured in metres, masses in kilograms, time in seconds.

2. "centimere - gram - second" system (c.g.s.)

In theis system lengths are measured in centimetres. masses in gram. time in seconds.

System	time	mass	velocity	accele	force ration	work or	power kinetic energy
m.k.s.	sec	kg	m/sec	m/sec^2	newton	newton. metre (joule)	newton m/sec (watt)
c.g.s	sec	gm	cm/sec	m/sec^2	dyne	dyne. cm (erg)	erg/sec

N.B.

When using any of the two above systems, the student has to use the suitable units of the system he chooses.

Example (1) :

A body of mass 1.5 kg is moving with velocity 12 m/sec. Find the value of its kinetic energy measured in units of newton metre and the erg.

Solution :

Since the mass is measured in kg and the velocity in m/sec, then the kinetic energy is measured in units of newton. metre (or joule).

$$T = \frac{1}{2} \times 1.5 \times (12)^2 \text{ newton. metre}$$
$$= 108 \quad \text{newton. metre (or joules)}$$
$$= 108 \times 10^7 = 1.08 \times 10^9 \text{ ergs}$$

Example (2) :

If the kinetic energy of the shot of a gun at a certain instant is equal to 22500 joules find, the magnitude of its velocity given that its mass is $\frac{1}{2}$ kg.

Solution :

$$\therefore \quad T = \frac{1}{2} m \|\overrightarrow{v}\|^2$$
$$\therefore \quad 22500 = \frac{1}{2} \times \frac{1}{2} \|\overrightarrow{v}\|^2$$
$$\therefore \quad \|\overrightarrow{v}\|^2 = \sqrt{90000} = 300 \text{ m/sec.}$$

Example (3) :

A body of mass 250 gm is moving with velocity 2 m/sec. Find its kinetic energy.

Solution :

In this example we have either to change the mass to units of kilogram and we then obtain the kinetic energy measured in units of newton. metre (or joule) or change the velocity to units of cm/sec and thus obtain the kinetic energy in units of erg.

If we choose the first way, then

$m \quad = \quad 250 \text{ mg} = 0.25 \text{ kg}.$

kinetic energy : $T = \frac{1}{2} \times 0.25 \times (2)^2 = 0.5$ joule.

If we choose the second way, then

$v \quad = \quad 2 \text{ m/sec} = 200 \text{ cm/sec}$

$\therefore \quad T = \frac{1}{2} \times 250 \times (200)^2 = 0.5 \times 10^7$ ergs

$= 0.5$ joule.

It is clear that the two methods are equivalent, as it is expected.

Example (4) :

A body of mass 800 gm is moving with a velocity given by the relation $\vec{v} = 12\,\hat{i} + 5\,\hat{j}$ where $\hat{i}$, $\hat{j}$ are two constant perpendicular unit vectors.

Find the kinetic energy of this particle given that the magnitude of the velocity is measured in units of cm/sec.

Solution :

Since the mass is measured in units of gram and the velocity in units of cm/sec, the kinetic energy will be measured in units of erg.

$\|\overrightarrow{v}\|^2 = (12)^2 + (5)^2 = 144 + 25 = 169$

kinetic energy : $T = \frac{1}{2} m \|\overrightarrow{v}\| \qquad = \frac{1}{2} \times 800 \times 169$

$= 67600$ ergs

Example (5) :

A body of mass 200 gm is projected wth velocity 200 cm/sec upwards along a line of greatest slope of a smooth inclined plane whose inclination to the horizontal is 30° Find its kinetic energy after 3 seconds from the instant of projection, and the change in its kinetic energy.

Solution :

Let $\hat{i}$ be a unit vector parallel to a line of greatest slope of the plane upwards. fig. (74).

Let v_0, v, a, be the algebraic measures (relative to the unit vector $\hat{i}$) of the initial velocity, velocity after 3 seconds, and acceleration vectors respectively.

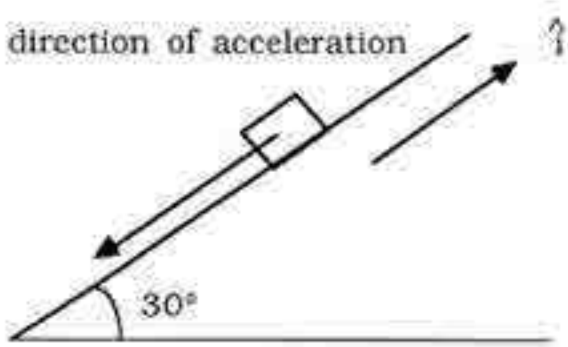


Fig. (74)

We know that.

$a = -g \sin 30^\circ = -980 \times \frac{1}{2} = -490$ cm/sec^2

Also $v_o = 200$ cm/sec

to find v we use the law $v = v_o + at$

$\therefore$ $v = 200 + (-490) \times 3 = -1270$ cm/sec

This shows that the body is moving at this instant downwards (opposite to the direction of the unit vector $\hat{i}$).

$T = \frac{1}{2} mv^2 = \frac{1}{2} (200) \times (-1270)^2 = 1.6 \times 10^8$ ergs

To calculate the change in kinetic energy, we have to calculate its initial kinetic energy.

$T_o = \frac{1}{2} m v_o^2 = \frac{1}{2} (200) \times (200)^2$

$= 4 \times 10^6$ ergs $= 0.04 \times 10^8$ ergs

Change in kinetic energy $= T - T_o = 1.56 \times 10^8$ ergs

Exercises (4 - 3)

1. Find the kinetic energy of a shot of mass $\frac{1}{4}$ kg and moving with velocity 300 m/sec.
2. Find the kinetic energy of a body of mass 50 gm and moving with velocity 20 m/sec.
3. Find the kinetic energy of a body of mass 2 kg and moving with velocity 25 cm/sec.
4. Find the velocity of a car of mass 1.5 tons if its kinetic energy is equal to 168750 joules.
5. Compare the kinetic energies of a bullet of mass 50 gm moving with velocity 300 m/sec and a locomotive of mass 48.6 tons and moving with velocity 1 km/h.
6. A body of mass 200 gm is moving with velocity $\vec{v} = 30\hat{i} + 40\hat{j}$ where $\hat{i}$, $\hat{j}$ are two perpendicular unit vectors. the magnitude of velocity is measured in units of cm/sec. Find the kinetic energy of this body.
7. A shell of mass 3 kg is fired from a gun with velocity $\vec{v} = -2000\hat{i} + 2000\hat{j}$ where $\hat{i}$, $\hat{j}$ are two perpendicular unit vectors, the magnitude of the velocity being measured in units of cm/sec. Find the kinetic energy of the shell at the instant of firing.
8. A body is moving with velocity $\vec{v} = 500\hat{i} + 100\hat{j}$ where $\hat{i}$, $\hat{j}$ are two perpendicular unit vectors. the magnitude of the velocity

being measured in units of cm/sec. Find the mass of this body f its kinetic energy is equal to 3.9 joules.

9. A particle of mass 50 gm falls from a point at a height of 10 metres meters from the surface of the earth. Calculate the kinetic energy of the particle when it is about to hit the ground.

10. A particle of mass $\frac{1}{2}$ kg is projected vertically upwards with velocity 14.7 m/sec from a point on the earth's surface. Calculate the kinetic energy of this particle after 1 second and after 1.5 second from the instant of projection.

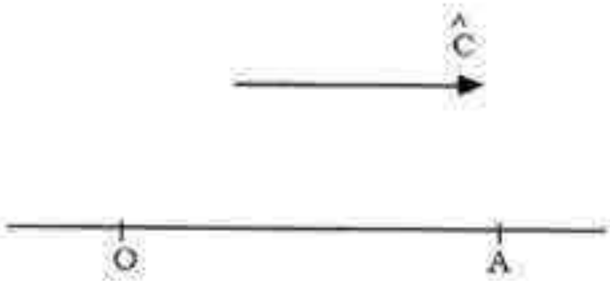
11. A particle of mass 200 gm starts motion from rest from the top of a smooth inclined plane whose length is 25 metres and inclined at an angle whose sine is $\frac{1}{10}$ to the horizontal. Find the kinetic energy of this particle when it reaches the bottom of the plane.

12. A particle of mass 5 kg is projected upwards along a line of greatest slope to a smooth inclined plane whose inclination to the horizontal is an angle whose sine is $\frac{1}{10}$ with velocity 4 m/sec. Calculate the change in the kinetic energy of this particle after 1 second from the instant of projection, and when it returns to the point of projection.

Principle of Work and Energe :

We discussed in the previous articles the concepts of work and kinetic energy and in what follows we are goin to show the relation between them in the case of motion of a particle along a straight line.

Consider the motion of a particle along a straight line under the action of a force $\overrightarrow{F}$ parallel to this line and let O be the position of the particle at the initial instant t = O, A be the position of the particle at time t . W the work done by the force $\overrightarrow{F}$ during the motion from O to A i.e. the work done from the start of motionin to the instant of time t. Let T_0 be the initial kinetic energy of the particle, i.e. its kinetic energy at the position O. T the final kinetic energy of the particle i.e. its kinetic energy at the position A.



(Fig. 75)

Also let $\hat{c}$ be a unit vector parallel to the straight line along which motion occurs. v, a the algebraic measures of the velocity and acceleration vectors at the time instant t respectively.

F the algebraic measure of the force vector relative to the unit vector $\hat{c}$. We have

$$T = \frac{1}{2} mv^2$$

$$\frac{dT}{dt} = \frac{d}{dt}\left(\frac{1}{2} mv^2\right) = \frac{1}{2} m \frac{d}{dt}(v^2)$$

$$= \frac{1}{2} m \times 2v \frac{dv}{dt} = \left(m \frac{dv}{dt}\right) \times v$$

$$= (ma) \times v$$

From Newton's second law ma = F

$$\therefore \frac{dT}{dt} = F\,v$$

i.e. "The time rate of change of the kinetic energy of a particle is equal to the power of the force acting on it".

On the other hand, we know that $\frac{dw}{dt} = F\,v$

comparing this relation with equation (1) we get :

$$\frac{dT}{dt} = \frac{dw}{dt}$$

i.e. $\frac{d}{dt}(T - W) = 0$

Since the vanishing of the derivative with respect to the time of function (T-W) means that this function takes a constant value at all instants, then.

$$T - W = \text{constat} \qquad (2)$$

To determine the value of this constant, we consider the value of the function (T-W) at the initial instant $t = 0$ where $T = T_0$, $W = 0$ (the force has not done any work yet)

$\therefore\ T_0 = \text{constant}$

substituting with the value of the constant in (2)

$$T - W = T_0$$

or $\quad T - T_0 = W \qquad (3)$

The last relation expresses the "Principle of Work and Energy" which states that

"The change in kinetic energy of particle when moving from an initial position to a final position is equal to the work done by the force acting on it during the displacement between these two positions."

Notice in using relation (3) that the units of measure of kinetic energy must be the same units of measure of the work.

Result :

If a particle starts motion from a certain position, and it returns to the same position, then its final kinetic energy is euqal to its initial kinetic energy.

Proof :

Since the displacement is represented by the zero vector, then the work done is equal to zero also:

$$\therefore\ T - T_0 = 0$$

i.e. $\quad T = T_0 \quad$ Q.E.D.

N.B.

If we apply this result to the motion of particle projected vertically upwards under the gravitational acceleration(considered constant), then the kinetic energy of the particle at any position during moving upwards is equal to its kinetic energy at the same positioin during moving downwards, and thus the magitude of the velocity is the same in both cases.

Example (1) :

Use the law which gives the velocity in terms of the time in a uniformly accelerated motion to find the kinetic energy of the moving particle, and hence deduce that the time rate of change of its kinetic energy is equal to the power of the acting force.

Solution :

In the uniformly accelerated rectilinear motion the velocity is given as a function of time in the form $v = v_0 + at$

where t is the time, v, v_0, a are the algebraic measures of the velocity vector at time t, the initial velocity vector at t = 0. The acceleration vector respectively relative to a unit vector parallel to the straight line on which motion occurs.

$$\therefore \; T = \frac{1}{2} mv^2 = \frac{1}{2} m (v_0 + at)^2$$

$$\therefore \frac{dT}{dt} = m\,(v_0 + at) \times a$$
$$= ma \times (v_0 + at) = mav$$

but $ma = F$, where F is the algebraic measure of the force vector.

$$\therefore \frac{dT}{dt} = F\,v = \text{power of the force causing motion.}$$

Example (2) :

Use the principle of work and energy to find the relation between velocity and displacement in the case of a vertical projectile under the action of the constant gravitational acceleration.

Solution :

Let s, v, v_0, a, F be the algebraic measures of the displacement, final velocity, initial velocity, acceleration, and force vectors respectively.

Applying relation (1) to the motion between the initial and final positions :

$$T - T_0 = W$$
$$\therefore \frac{1}{2} mv^2 - \frac{1}{2} mv_0^{\,2} = Fs$$

but $F = ma$

$$\therefore \frac{1}{2} mv^2 - \frac{1}{2} mv_0^{\,2} = Fs$$

multiplying both sides of this equation by $\frac{2}{m}$ we get

$$v^2 - v_0^{\,2} = 2\,as$$

Which is the well known relation between velocity and displacement.

Example (3) :

A bullet of mass 200 gm is fired with velocity 400 m/sec at a thick barrier, and it embedded in it at a depth of 20 cm. Find the magnitude of the resistance force of the barrier's material to the motion of the bullet, it this force is considereed constant.

Solution :

Let A be the position of enetrance of the bullet in the barrier, B the position where it is embedded, $\overrightarrow{R}$ is the force of resistance measured in units of dyne.

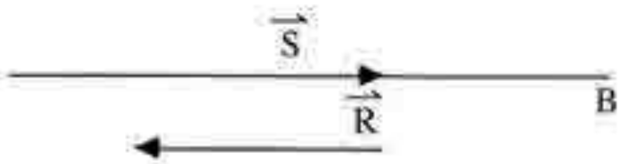


Fig. (76)

We have Ab = 20 cm.

Since the force of resistance is acting in a directioi opposite to the of displacement, fig. (76), then the work done by this force will be negative, and is calculated as follows :

$$W = -AB \times R = -20\ R$$

The kinetic energy of the bullet when it enters the barrier

$$T_A = \frac{1}{2} \times 200 \times (400 \times 100)^2 = 1.6 \times 10^{11} \text{ ergs}$$

(notice the change of velocity to units of cm/see)

The kinetic energy of the bullet at positions B :

$T_B = O$ since the bullet is at rest in this position.

Change in kinetic energy of bullet :

$T_B - T_A = 1.6 \times 10^{11}$ ergs

$\therefore \quad T_B - T_A = W$

$\therefore \quad -1.6 \times 10^{11} = -20\ R$

$\therefore \quad R = \dfrac{1.6 \times 10^{11}}{20} = 8 \times 10^{9}$ dynes

Example (4) :

A particle of mass 20 kg is let to descend along a line of greatest slope of a smooth inclined plane whose inclination to the horizontal is 30°. Using the principle of work and energy find the velocity of the particle when it has moved a distance of 5 metres.

Solution :

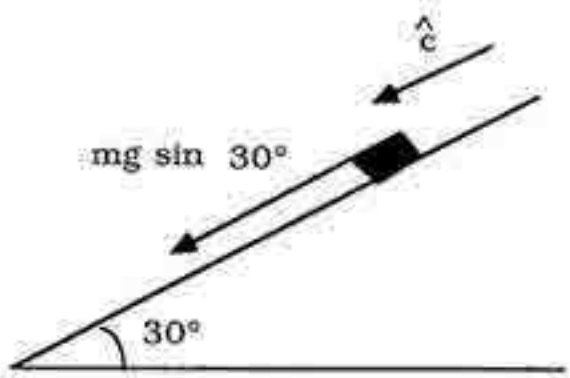


Fig. (77)

Choose a unit vector $\hat{c}$ in the direction of motion and let v be the algebraic measure of the velocity vector relative to the unit vector $\hat{c}$, L the distance described. The only force doing work is the component of the weight force parallel to the line of greatest slope on which motion occurs.

This force is directed down the plane and a magnitude mg sin 30°. Where m is the mass of the particle g the magnitude of the gravitional acceleration. fig. (77). The work done by this force during the given displacement

$W = (mg \sin 30^\circ) \times L$

$= (20 \times 9.8 \times \frac{1}{2}) \times 5$

490 newton. metre (or joules)

$\therefore \quad \frac{1}{2} mv^2 - o = W$

$\therefore \quad \frac{1}{2} \times 20 \times v^2 = 490$

$\therefore \quad v^2 = 49$

$\therefore \quad v = 7$ m/sec.

and this is the velocity with which the particle moves after it has described 5 metres from its initial position.

Example (5) :

A body of mass 300 gm is placed at the top of an inclined plane whose height is 1m. Find the velocity with which the body reaches the bottom of the plane if the work done by the resistance force of the plane to the motio is equal to 1.59 joules.

Solution :

Let L be the length of the plane measured in metres, θ the inclination of the plane to the horizontal. Two forces acting on the body in a direction parallel to the direction of motion, the component of weight acting in a line of greatest slope downwards and of magnitude

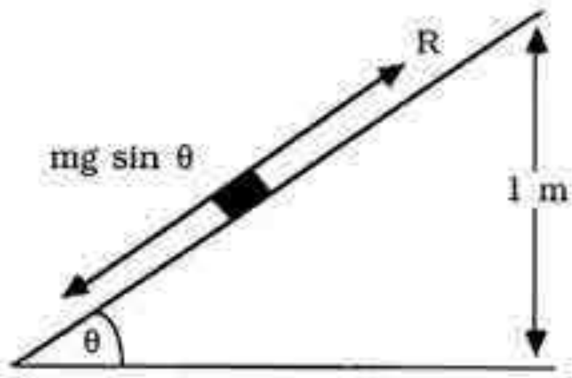


Fig. (78)

mg sin θ and the resistance force, let its magnitude be R and acting along a line of greatest slope upwards.

The work done during the motion of the body from the top of the plane to its base :

$$W = (mg \sin ø - R) \times L$$

$$= (0.3 \times 9.8 \times \frac{1}{L} - R) \times L = 0.3 \times 9.8 - RL$$

but RL = work done by resistance force

$$= 1.59 \text{ joules}$$

$\therefore \quad W = 0.3 \times 9.8 - 1.59 = 1.35$ joules

$\therefore \quad T - T_0 = W$

$\therefore \quad \frac{1}{2} mv^2 - 0 = 1.35$

$\therefore \quad u^2 = 9$

$\therefore \quad V = 3$ m/sec.

Exercises (4 - 4)

1. A particle of mass m is moving in a straight line so that the algebraic measure of its velocity vector is given in terms of the time in the form $v = A + Bt + Ct^2$. Find the kinetic energy of this particle, and prove that the power of the force causing the motion at the instant $t = 0$ was equal to m AB.

Use the principle of work and energy to solve the following exerciese.

2. A body of mass 1 kg is let to fall from a height 10 metres above the Earth's surface. Find its kinetic energy when it is about to hit the ground.

3. A particle of mass 200 gm is projected vertically upwards from a position on the ground with velocity 15 m/sec. What is its kinetic energy when it is at a height of 10.4 metres from the point of projection ?

4. A body of mass 400 gm is projected vertically downwards from a position 3 metres above the Earth's furface with velocity 5 m/sec. Find its intitial kientic energy, and its kinetic energy when it is about to hit the ground.

5. A bullet of mass 200 gm is fired with velocity 294 m/sec at a piece of wood, it is embedded in it at a depth of 20 cm. Find the resistance force of the wood to the motion of the bullet measured in units of kg. wt assuming that it is constant.

6. A bullet is fired horizontally with velocity 700 m/sec at a piece of wood. It is embedded in it a depth of 8 cm. If a similar bullet is fired with the same velocity at a fixed target made of the same wood of thickness 6 cm, what is the velocity with which the bullet comes out of the target assuming that the resistance is constant ?

7. A bullet is fired with velocity 300 m/sec at a wooden target and it is embedded in it a depth of 27 cm. What is the velocity with which a similar bullet should be fired at a target made of the same wood and of thickness 3 cm so that it is embedded in it, while it is about to penetrate it, assuming that the resistance is constant ?

8. A body of mass 500 grams moves with the velocity $\overrightarrow{U} = 3\hat{i} + 4\hat{j}$ where $\hat{i}$, $\hat{j}$ are two perpendicular unit vectors and the magnitude of the velocity is measured with the unit metre/sec. Determine the kinetic energy of this body in ergs.

9. A bullet is fired from a gun with velocity 800 m/sec at a thick wooden barrier, it is embedded in it at a depth 8 cm from the surface if another bullet is fired from the same gun at a barrier made from the same material as the first barrier and of thickness 6 cm and it penetrates it, find the velocity of the bullet at the instant it comes out of the barrier, given that the resistance of wood to the motion of the bullet is the same in both cases.

10. A body descends from rest along a line of greatest slope of a smooth inclined plane whose inclination to the horizontal is an angle whose tangent is $\frac{3}{4}$, a distance of 100 metres. Find the velocity of the body at the end of its path.

11. A body of mass 5 kg is pushed downwards with velocity 20 cm/sec along a line of greatest slope of a smooth inclined plane of length 400 cm and height 150 cm. Find the kinetic energy of this body when it reaches the bottom of the plane.

12. A body of mass 200 gm is placed at the top of an inclined plane whose height is 1 metre, and it descends until it reaches the bottom of the plane with velocity 2 m/sec. What is the work done by the resistance force given that it is constant ?

13. A body of mass 200 kg descends from rest along a line of greatest slope of an inclined plane whose length is 16 metres and whose height is 5 metres. If the resistance to the motion of the body is equal to 1/4 of its weight. Find the kinetic energy of the body when it reaches the bottom of the plane.

14. A horizontal force of 10 kg. wt acted for 5 seconds on a body of mass 20 kg. It moved from rest on a horizontal plane. Find the velocity of the body at the end of the interval. If it struct a body at rest, of mass 20 kg to form a body, find their common velocity. Calculate the kinetic energy lost and explain what happened to this loss in K.E ?

15. A force of 5 kg. wt acts on a mass of 196 kg which moves in a straight line in the direction of the force. It convered a distance 280 cm. Calculate the increase in the kinetic energy in ergs. If the kinetic energy of the mass at the end of the distance is 1411.2 million ergs Calculate the velocity of the mass at the insant in which the force acts.

16. Two spheres of masses 30 gm and 90 gm move on one straight line on a smooth horizontal table and in two opposite directioins

with velocites of magnitudes 50 cm/sec and 30 cm/sec respectively. If the two spheres formed one body after impact, calculate the velocity of the body and the los kinetic energy by impact.

17. A sphere of mass 100 gm falls from a height of 3.6 metres on horizontal ground, and it collides the ground and rebound vertically upwards. If the loss in the kinetic energy after collision by the ground is 1.96 joules, find the distance which the sphere rebound after collision.

18. A hummer of mass 1ton falls from a height of 4.9 metres vertically on a pole of mass 400 kg, and penetrates it in the ground a distance of 10 cm. Determine the common velocity of the hummer and the pole just after impact. Determine also the lost kinetic energy by impact and the resistance of the ground.

Potential Energy :

Consider the motion of a particle along a straght line under the action of a constant force $\vec{F}$ parrallel to this line, let O be the position of the particle at the initial instant t = 0, A its position at the final instant t, T_0, T the kientic energies at those two positios respectively, W the work done by the force $\vec{F}$ in moving the body from position O to position A fig. (79)

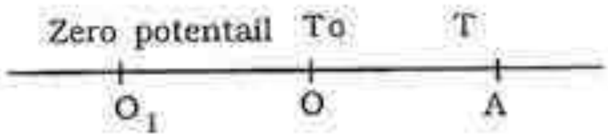


Fig. (79)

Definition :

The potential energy of a particle at a certain instant, denoted by P is defined as the work done by the acting force on the body if it moved it from its position at this instant to a fixed position on the straight line on which motion occurs.

To deduce a mathematical expression for the potential energy. Let us choose a fixed position 0_1 on the straight line on which motion occurs. Let P be the potential energy at position A.

From the definition $P = \vec{F} \cdot \overrightarrow{AO_1}$.

It is clear from the definition of potential energy that :

1) Potential energy at the fixed point O_1 is zero, thus point O_1 is called the "Point of zero potential energy".

2) If we change the point of zero potential energy, the potential energy changes, also, let P_0 be the potential energy of the particle at the initial position O.

$$\therefore \quad P_0 = \overrightarrow{F} \odot \overrightarrow{AO_1}$$

$$\therefore \quad P - P_0 = \overrightarrow{F} \odot \overrightarrow{AO_1} - \overrightarrow{F} \odot \overrightarrow{OO_1}$$

$$= - \overrightarrow{F} \odot (\overrightarrow{OO_1} - \overrightarrow{AO_1})$$

$$= - \overrightarrow{F} \odot (\overrightarrow{OO_1} + \overrightarrow{A_1O}) = - \overrightarrow{F} \odot \overrightarrow{OA}$$

But $\overrightarrow{F} . \overrightarrow{OA} = W$ is the work done by the force $\overrightarrow{F}$ in moving the particle from the intitial position O to the final position A.

$$\therefore \quad P - P_0 = -W$$

This means that

The change in the potential energy of a particle when it moves from an initial position to a final position is equal to minus the work done by the force during the motion.

On the other hand, we know that the principle of work and energy states that :

$$T - T_0 = -W$$

Comparing the last two relations, we find

$$T - T_0 = - (P - P_0)$$

i.e.

$$T + P = T_0 + P_0$$

Which is another form of the principle of work and energy. Since position A is a general position of the particle, then this relatioin states that :

The sum of the kinetic and potential energies is constant during motion".

Unitis of measure of potential energy :

It is clear that units of measure of potential energy are the same as the units of measure of work and kinetic energy.

N.B.

When studying the vertical motion of a projectile under the action of the ocnstant gravitational acceleration, we agree to choose the point of zero potential energy at the earth's surface.

Example (1) :

Calculate the potential energy of a projectile of mass ma moving vertically under the constant gravitational acceleration.

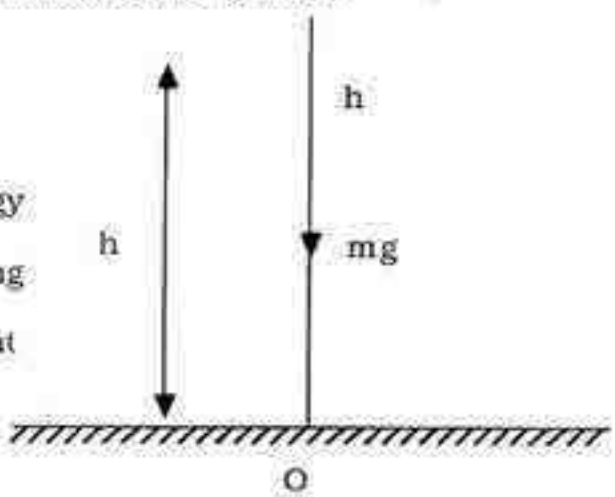


Fig. (80)

Solution :

Let A be a general position of the particle at height h from the, earth's surface, $\hat{j}$ a unit vector dertically upwards. Choose the point of zero potential energy O_1 at the point of intersection f the vertical through the projectile and the horizontal plane of the ground (Fig.80).

The force acting on the particle is the weight.

$\vec{F} = -mg\,\hat{j}$, g is the gravitational acceleration

If P is the potential energy at A, then

$$P = \vec{F} \odot \overrightarrow{AO_1}$$

but $\overrightarrow{AO_1} = -h\hat{j}$

$\therefore \quad P = (-mg\,\hat{j}) \odot (-h\hat{j})$

$\therefore \quad P = mg\,h$

i.e. "The potential energy of a vertical projectile at any positioin is equal to the product of the weight of the particle times the height of this position above the earth's furface".

Example (2) :

A rocket is projected vertically upwards from a position on the surface of the earth with velocity 1400 m/sec. It hits a target at t a height of 2000 metres above the surface of the earth.

What is the velocity of the rocket when the target is destroyed ?

Solution :

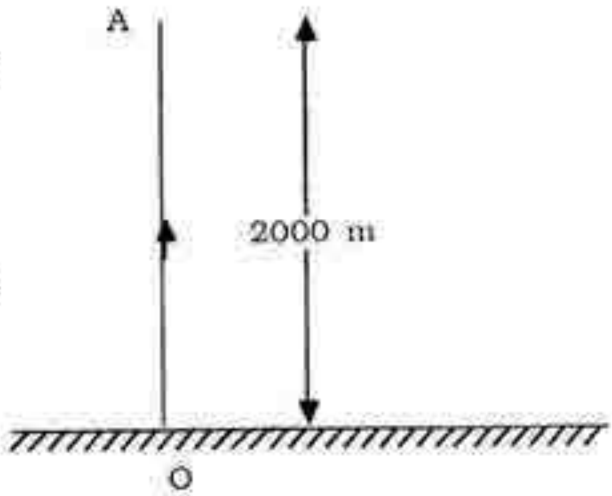


Fig. (81)

Potential energy at the point of projection O

$P_0 = 0$

Potential energy at the position of the target A

$P = mgh$

Where m is the mass of the reocket measured in kilograms, g the gravitational acceleration measured in units of m/sec^2, h = 2000 m. According to the principle of work and energy :

$$\frac{1}{2}mv^2 + P = \frac{1}{2}mv_0^2 + P_0$$

$$\frac{1}{2}mv^2 + mgh = \frac{1}{2}mv_0^2 + 0$$

$$\therefore \quad V^2 = v_0^2 - 2gh$$

$$= (1400)^2 - 2 \times 9.8 \times 2000 = 19208 \times 10^2$$

$$\therefore \quad V \simeq 1386 \text{ m/sec.}$$

Example (3) :

Find the potential energy of a particle moving along a line of greatest slope of a smooth inclined plane whose inclination to the horizontal is an angle of measure α under the action of the constant gravitational acceleration.

Solution :

Let A be a general position of the particle on the plane, and h the height of this positio above the surface of the Earth, $\hat{c}$ a unit vector parallel to a line of greatest slope and directed downwards. Choose the point of zero potential energy at O_1 the point of intersection of the line of greatest slope of the plane with the horizontal ground as in fig. (82).

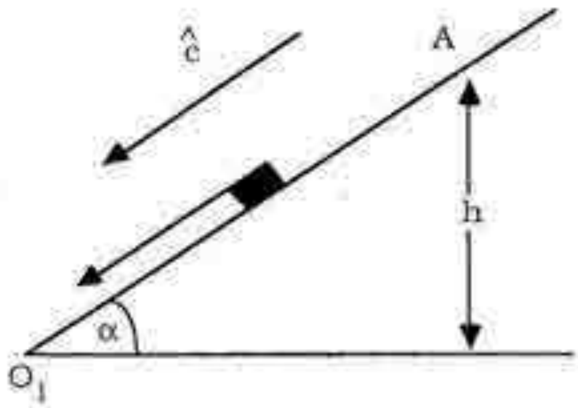


Fig. (82)

The only force parallel to the directio of motion is the component of the weight force acting along a line of greatest slope downwards, and of magnitude mg sin α where g is the gravitational acceleration, m is the mass of the particle.

$$\vec{F} = (mg \sin \alpha)\ \hat{c}$$

If P is the potential energy at A, then

$$P = \vec{F} \odot \overrightarrow{AO_1}$$

but $\overrightarrow{AO_1} = AO_1\ \hat{c}$.

$$\therefore \quad P = (mg \sin \alpha)\ \hat{c} \odot \left(\frac{h}{\sin \alpha}\right)\ \hat{c}$$

$$= mgh\ (\hat{c} \odot \hat{c})$$

$\therefore \quad P = mgh$

i.e.

The potential energy of a particle moving along a line of greates slope of a smooth inclined plane at any position is equal to the product of the weight of the particle times the height of this positioin above the surface f the earth"

Example (4) :

A body of mass 4 kg. moved up 75 cm along the line of greatest slope of a smooth inclined plane whose inclination to the horizontal is an angle of measure 30° . Find the increase in its potential energy in joules.

Solution :

The increase in the potential energy =

$p - p_0 \ = mgh - 0$

$= 4 \times 9.8 \times 0.75 \sin 30^\circ$

$= 14.7$ joules

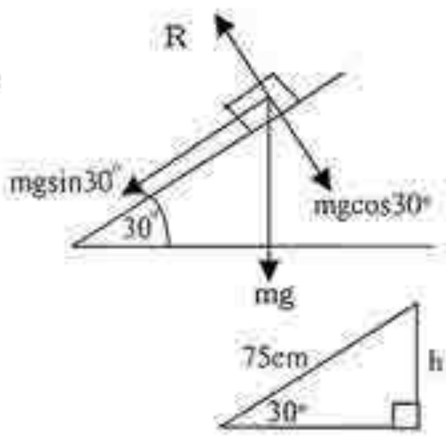


Fig. (83)

Exercises (4 - 5)

1. Calculate the potential energy fo a body of mass 450 gm at a height of 30 metres above the surface of the earth, giving your answer measured in joules.
2. Find the potential energy of a body of mass 3 kg at a height of 20 cm above the surface of the Earth, giving your answer measured in ergs.
3. A helicopter whose weight is 3500 kg.wt decends vertically from a height of 150 metres to a height of 50 metres from the surface of the Earth. What is the loss in its potential energy ?
4. A winch lifts a weight of 150 kg. wt vertically from its position on the ground to a new position at a height of 6 metres from the surface of the Earth. What is the increase in the potential energy of the body.
5. A particle of mass 250 gm descends a distance of 80 cm along a line of greatest slope of a smooth inclined plane whose inclination to the horizontal is an angle of measure 60°. What is the change in its kinetic energy ?
6. A body whose weight is 2 kg. wt ascends a distance of 120 cm along a line of greatest slope of a smooth inclined plane of inclination 30° to the horizontal. Calculate the increase in its potential energy.

7. A body of mass 5 kg is placed at a height of 15 metres from the surface of the ground. Find its potential energy. If it falls vertically, find its potential energy and kinetic energy when it is 5 metres above the surface of earth.

8. A simple pendulum consists of a light string of lenght 90 cm carrying in its end a mass of 75 gm and moves through an angle 120°. Fing :

   i) the increase in its potential energy at the end of its motion than its potential energy half way of its motion.

   ii) the velocity of the mass at half way of its motion.

9. A body moved under the action of the force $\overrightarrow{F} = 6\hat{i} + 2\hat{j}$ from the position A to the position B in two minutes. If the position vector of the body is given by the relation :
$\overrightarrow{r} = (3t^2 + 2)\hat{i} + (2t^2 + 1)\hat{j}$ find the change in the potential energy of the body where the magnitude of $\overrightarrow{F}$ is measured in newtons and the magnitude of $\overrightarrow{r}$ s measured in metres and t in seconds.

10. A ring of mass $\frac{1}{2}$ kg slides upon a vertical cylinder pole. If its velocity is 6.3 metre/sec after it covered a distance of 4.8 metres from the beginning its motion, calculate-using the principle of work and energy - the work done by the resistance during the motion.

Model Tests

- Model One -

(1) a) Find the resultant of the two parallel forces of magnitude 70 newtons, 30 newtons and the distance between their lines of action is 50 cm in the following two cases:

1^{st} : The two forces in the same direction.

2^{nd}: The two forces in opposite directions.

[100 newtons 15 cm from the first],

[40 newtons 37.5 cm from the first]

b) Define coefficient of friction:

A body of weight w is placed on a plane inclined to the horizontal with an angle 30°, the coefficient of friction between them is $\frac{1}{3}$. Find the magnitude of the least horizontal force directed towards the plane and its line action lies in a vertical plane passing through the line of greatest slope of the plane and acting on the body and makes it about to slide :

a) down the inclined plane.

b) up-ward the inclined plane.

$$\left(\frac{5\sqrt{3}-6W}{13} \;,\; \frac{2\sqrt{3}\,W+6}{13} \right)$$

(2) a) If the force $\overrightarrow{F} = L\hat{i} - 5\hat{j}$ act at the point A (6 , 3) and the moment vector of $\overrightarrow{F}$ about the point B (8 , -1) equals -2 K find the value of the constant L.

b) Write the necessary and sufficient conditions for equilibrium of a set of forces.

A uniform rod $\overline{AB}$ of length L is in equilibrium in a vertical plane such that its end A rests on a vertical smooth wall, and its other end B rests on a horizontal floor and coefficient of friction between them is $\frac{1}{3}$. If the

measure of the angle of inclination of the rod is given by $\tan\ \theta = \frac{1}{2}$,
what is the furthest point from B, on the rod from which a weight is double that of the rod may be hung such that the rod is about to slip.

[The point at a distance $\frac{3}{4}$ L from B]

(3) a) Define : couple

$\overline{AB}$ is a uniform rod of length 50 cm. and weight 10kg.wt. acts at its midpoint, it can easily rotate in a vertical plane about a fixed nail at point A, a couple of norm moment 12kg.wt.cm. in a vertical plane acted on the rod. In the state of equilibrium of rod. Prove that reaction of the nail at A equals the weight of the rod find the inclination of the rod to the horizon in case of equilibrium.

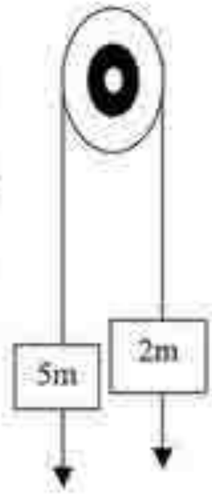
b) $\overline{AB}$ is a wooden beam of length 20 meters and weight 50.kg.wt. acts at its midpoint, rests horizontally on the supports one of them is at a distance 2 meters from A and the other is at a distance 5 meters from B. A man of weight 70kg.wt. moves on the beam from A towards B find:

1) Reactions of two supports when the man stands at A.

2) The maximum distance which the man can move such that the beam doesn't over turn.

[R_1 = 100kg.wt., R_2 = 20kg.wt., maximum distance = 20 meters]

(4) a) In the oppsite figure :



Two masses 5m, 2m are connected from the ends of a light string passes over a smooth pulley. The system is kept in equilibrium when the two parts of the string are vertically. If the system is allowed to move from rest. Find acceleration of the system. and if pressure on the pulley = 112 newtons find the value of m .

[$a = 4.2$ m/sec^2 , m = 2kg.]

b) Find the reaction of a lift in kilogram weight on a person whose mass is 73.5 kg. in the follwoing cases:

1) If the lift is at rest.
2) If the lift is moving upwards with a uniform acceleration of mgnitude 140 cm/sec^2.
3) If the lift is moving downwards with a uniform acceleration of magnitude 140cm/sec^2 [73.5, 84, 63kg.wt.]

(5) a) Define : The Power

A car of mass 2 tons, the power of its motor is 20 horses. Is moving along a horizontal to the magnitude of the velocity. If the maximum velocity of the car on the road is 90 kg. m/h. what is the magnitude of the resistance force for each ton of the mass of the car when it moves with velocity 18 km/h. Calculate the momentum of the car at this velocity.

[6kg.wt. / ton, 10000 newton.sec]

b) A particle of mass 200gm is projected along the line of greatest slop of a smooth inclined plane whose inclintion to the horizontal is an angle whose sine is $\frac{8}{49}$ with velocity 30cm/sec. Calculate the change in the ptential energy of this particle when its velocity becomes 18cm/sec.

[57600 ergs]

(6) a) A body of mass 2 Kg with displacement vector $\overrightarrow{S} = 4t^2\ \hat{i} + 3t\ \hat{j}$ moves under the action of a force $\overrightarrow{F}$. Find the work done by the force after two seconds from the start of motion. knowing that S is measured in meters ,F in Newton and t in seconds.

b) A sphere of mass 290 gm moves along a straight line with velocity of magnitude 32 cm/sec, collides with another sphere of mass 550 gm at rest. If the two spheres moves as one body after impact,

find: (i) The common velocity of them after impact.

(ii) The kinetic energy lost due to impact .

(iii) The needed resistance to make the body comes to rest after it covers 20 cm from the position of impact .

- Model Two -

(1) a) If $\overrightarrow{OX}$, $\overrightarrow{OY}$ are two perpendicular directions. $\hat{i}$, $\hat{j}$ are two unit vectors in these two directions respectively. The force $\overrightarrow{F} = 3\hat{i} - 4\hat{j}$ acts at the point A (1 , 2). Calculate the moment of this force about the point O, and hence find the length of the perpendicular from O to the line of action of the force.

[$-10\hat{K}$, 2 units of length]

b) Define : Angle of friction

A body of weight 60 newton rests on a rough horizontal plane. Two forces of magnitude 20, 40 newton act horizontally on the body. the angle between their lines of actions = 120°. If the body is in equilibrium, prove that the angle of friction λ is not less than 30°. If $\lambda = 45^\circ$ and the two forces act in their previous directions and the 40 newtons force remains constant, find the least value for the other force to move the body and determin the direction of motion.

($(20 + 20\sqrt{6})$ newtons)

(2) a) $\overline{AB}$ is a uniform rod of length 150cm, and of weight 100 newtons. It rests in a horizontal position on two supports, one of them A and the other at a point C distant 25cm from B . Find the pressure on each support, and find the magnitude of the weight that must be suspeneded at B in order that the rod will be on the point of rotatioin. What is the magnitude of the pressure on the support at this instant?

[40.60 newtons, 200, 300 newtons]

b) ABCD is a rectangle in which AB = 30 cm , BC = 40 cm . Forces of magnitudes 12 , 24 , 12 , 24 Kg.wt act along $\overrightarrow{BA}$, $\overrightarrow{BC}$, $\overrightarrow{DC}$, $\overrightarrow{DA}$ recpectively . Prove that this system equivalent to a couple and find magnitude of its moment hence find two forces acting at A , C and parallel to $\overleftrightarrow{BD}$, such that the system is in equilibrium.

(240 kg. wt, 5, 5 kg.wt)

(3) a) Two parallel forces having the same direction act at the points A , B of a rigid body , and their magnitudes are 4 , 7 newtons respectively . If the force of magnitude 7 newton is transfared parallel to itself a distance L to another point on the ray $\overrightarrow{BA}$ prove that the point of action of the resultant transferes a distance of $\frac{7}{11}$ L.

b) A uniform ladder rests in a vertical plane with its ends on a horizontal floor and aginst a vertical wall. The ladder is incilined to the wall with an angle which its tangent is $\frac{6}{11}$, and the coefficient of friction between the ladder and either the wall or the floor is $\frac{1}{3}$. If the man whose weight is three tims as that of the ladder is ascending it, prove that he cannot exceed seven-tenths of the length of the ladder whitout disturbing eqilibrium.

(4) a) A force $\overrightarrow{F} = 2\,\hat{i} + 3\,\hat{j}$ acts on a particle so its position vector $\overrightarrow{r}$ given at any instance t by the relation

$\overrightarrow{r} = (3t^2 + 7)\,\hat{i} + (t + 2)\,\hat{j}$, where F is in newtons , the distance in meter, and the time t in seconds . Calculate the work done by the force $\overrightarrow{F}$ from

t = 1 sec to t = 3 sec.

b) A car of mass one ton moves with a velocity of magnitude 54 km/h on a horizontal road. Find the power of the engine if the engine if the resistance is 30 kg.wt. and if the power of the engine the resistance did not change, find the constant velocity by which the car ascends an incilined plane of inclination θ with the horizontal where sine θ is $\frac{1}{20}$.

[6 hours 20, 25 km./h.]

(5) a) A hammer of mass 540 kg falls vertically from a hight of 2.5 meters on a foundation pole of mass 60 kg. to insert it in the ground a distance of 9 cm, if both the hammer and the pole move after impact as one body . Find the resistanc of the ground assuming that it is constant.

b) A body of mass 120 gm rests on a rough plane inclined to the horizontal by an angle whose tangent is $\frac{3}{4}$, The body is connected, by a string which passes over a smooth pulley fixed at the top of the plane, to a mass of 160gm. If the coefficient of friction is $\frac{2}{3}$, and the system starts its motion from rest, determine the distance traversed in three seconds.

(252 cm)

(6) a) A body of weight $\frac{1}{2}$ kg fell vertically downward in a well, it reached to the surface of the water after one second then it penetrated into the water till it reached to its bottom after one second more . find the change in the momentum from the instant of reaching the surface of the water till to strikes its bottom.

b) A body of mass 20 kg is plased on smooth plane inclined at an angle θ to the horizontal where $\sin \theta = \frac{1}{20}$, if a force of 16 kg. wt acted upon along the line of the greatest slope upwards. Find the magnitude of the acceleration if the effecting of the force is stopped after 3 seconds from starting the motion, find the distance covered by the body before it comes to rest instantoneously.

- Model Three -

(1) a) Four parallel forces having the same direction and of magnotudes 1, 2, 3, 4 kg.wt. act at the points A, B, C, D respectively which line on the same straight line which is perpendicular to the lines of actions of the forces. Find the resultant of these forces given that AB = 30 cm, BC = 40 cm , CD = 50 cm

[10 kg.wt. , 75 cm. from A]

b) ABCDEF is a regular hexagon , the length of whose side is 8cm. Forces of magnitude 8 , 1 , 8 , 6 , 5 gm.wt act along $\overrightarrow{AB}$, $\overrightarrow{DC}$, $\overrightarrow{BE}$ and $\overrightarrow{FA}$ respectively.

(i) Prove that the set of forces equivalent to a couple and find the magnitude of its moment.

(ii) Find the magnitude and the direction of the two forces which act perpendicular to $\overline{AD}$ such that the system in equilibrium.

(2) a) $\overline{AB}$ is a uniform rod of weight 50 newton, length 160 cm. is suspended by two vertical strings from two points C, D where AC = BD = 40 cm. , a weight of magnitude 10 newton is suspended from B. Find the magnitude of the weght that should be suspended from A to keep the rod horizontally where the magnitude of the tention at C is twice that at D.

[24 newton, T = 24 newton]

b) A body of weight 10 newton is placed on an incilined rough plane of angle whose tangent is $\frac{4}{3}$ to the horizontal. Find the value of the least and greatest force which acting on the body about along the line of great slope upwards to make the body about to move (Given that the coefficient of the friction between the plane and the body equals $\frac{1}{2}$)

[5 , 11 newtons]

(3) a) ABCD is a rectangle in which AB = 8 cm , BC = 6 cm . Forces of magnitudes 12 , 10 , F , K newtons act along $\overrightarrow{AB}$, $\overrightarrow{CB}$, $\overrightarrow{CD}$, $\overrightarrow{AD}$ recpectively . If the algebraic sum of the moment of these forces about each of the two points C and M vanishes (When M is the center of the rectangle) . Find the values of F and K .

[$\frac{40}{3}$, 9 N]

b) AB is a uniform ladder of length 520 cm and weight 24 kg .wt rests with its end B aganist a smooth vertical wall and with its end A on a smooth horizontal plane . The ladder is kept from slipping by a string , one end of string fixed vertically below B and the other end fixed to one stair at a distance 130 cm from A. If the end B is at a distance 480 cm from the horizontal plane. find the reaction of the ground at its ends A , B and find the magniyude of the tension in the string .

[30 , 7.5 , $\frac{3\sqrt{41}}{2}$ newton]

(4) a) A man of mass 70 kg.wt stand on the floor of a lift , if the lift move downwards with a uniform raterdation magnitude 49 cm/sec^2 find the tention of the rope and the pressure of the man on the floor of the lift.

b) A body was left to slides along an inclined plane which ends by a horizontal plane with the same rough , if the coefficient of friction equals $\frac{1}{4}$, the length of inclined plane is 9 meters and the inclined plane inclines by angle of measure θ where tan
$\theta = \frac{3}{5}$. Find the maximum distance covered by the body on the horizontal plane befor come to rest. given that the velocity of the body does not change in the two planes.

(5) a) A body of mass 400 gm is placed on a horizontal smooth table and tied to a light string passing on a smooth pulley at the edge of table and the other end of the string is tied to a body of mass m gm. If the tension in the string is 80gm.w, find :

a) The paressyre on the pulley. [280 $\sqrt{2}$ gm.wt.]

b) The acceleration of the system. [196 cm/sec^2]

c) The value of m. [100 gm.]

b) A lorry of mass 2 tons and the power of its motor is 20 horses is moving alonge a horizontal road with its maximum velocity of magnitude 80 kg/h. Find the magnitude of the road resistance to the motion of the lorry. If this lorry is loaded with a weight of magnitud 475 kg.wt, and it moves upwards alonge an incilined road whose incilination to the horizontal is an angle of sine $\frac{1}{15}$, what is its maximum velocity alonge this road, given that the magnitude of the resistsnce of the incilined road is twice the magnitude of the resistance along the horizontal road.

[67.5 kg.wt, 18 km/h]

(6) a) A man of weight 72 kg.wt ascends along an inclined road whose inclination to the horizontal is an angle of sine $\frac{1}{4}$ and he covered a distance of 100 meters .Find the change in the potential energy of this man.

b) A force of magnitude 48 gm.wt acting on a body which placed on a horizontal plane for interval of certain time at the end of this time interval the body gains kinitic energy of magnitude 18900 gm.wt and the momentum of the body at this moment equals 176400 gm.cm/sec. Find the mass of the body and the resistance of the plane assuming that it is constant. find also the time of force effect.

- Model Four -

(1) a) Prove that : sum of moments of a set of coplanar forces act a point with respect to any point in space equal moment of their resultant about the same point.

Forces $\overrightarrow{F_1} = \hat{i} + \hat{j}$, $\overrightarrow{F_2} = \hat{i} - \hat{j}$, $\overrightarrow{F_3} = 2\hat{i} - 3\hat{j}$ act at the point A (1 , 1). Find the moment of each force about the origin and hence find the length of the perpendicular from from the origin to the line of action of the resultant of these forces.

[$\overrightarrow{o}$, $-2\hat{K}$, $-5\hat{K}$, $\frac{7}{5}$]

b) A body of weight 32 kg.wt is about to move under its own weight when placed on a rough plane which inclined to the horizontal with an angle whose tangent $\frac{1}{4}$. If the body is placed on a horizontal plane which is as rough as the inclined plane and is acted on by an upward pull in direction inclined to the horizontal with an angle whose sine is $\frac{4}{5}$ so that the motion is about to move. Find the magnitude of this force and the normal reaction.

[10 , 24 kg.wt]

(2) a) ABCD is an un-uniform rod whose length is 35 cm. it rests in a horizonal postion on two supports B, C wher AB = 6 cm. CD = 7 cm. If a mass of 120 gm in suspended at A or a mass of 180 gm. is suspended at D, the rod is about to rotate. Find the weight of rod and the distance between its point of action and A .

[8 cm, 90 gm.wt.]

b) $\overline{AB}$ is a uniform rod of length 120 cm and weight 6 Kg.wt, its end A is connected to a hinge on a vertical wall , a body of weight 8 kg.wt is suspended from a point on the rod at a distance 30 cm from its end A if the rod is kept in equilibrium horizontally by means of string , one of end is tied to end B of therod and the other end of the string is to a point on the wall vertically above A , if the string is inclined by angle of measure 30° to the horizontal find:

(i) the magnitude of the tension in the string.

(ii) the magnitude of the reaction of the hinge.

(3) a) A uniform rod of length 100 cm and weight 8 Kg.wt is suspended from two points each of them at a distance 10 cm from its ends by two vertical string such that the rod in horizontal position. If a weight of magnitude W at a distance 20 cm from the mid of the rod , find the magnitude of W . Given that the greatest tension at each string does not exceed 16 kg.wt.

b) ABCD is a trapezium in which AB = 12 cm , CD = 6 cm , DA = 8 cm and m ($\angle$ A) = m ($\angle$ D) = 90°. Forces of magnitude 12, 18, 15, 9 newtons act along $\overrightarrow{DA}$, $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{CD}$ recpectively.

(i) Prove that these forces equivalent a couple.

(ii) Find norm of its moment.

(4) a) A bullet of mass 12 gm is fired with velocity 21 m/sec. Find the kinetic energy normally with a vertical wall and it penetrates in it a distance of 6 cm, find the resistance of the wall to the bullet measured in kilogram weight, assuming that it is constant.

[2.646 joules , 4.5 kg.wt.]

b) A rough plane is incilined to the horizontal at an angle 30° and is joined at its top to another rough horizontal plane. A body of mass 60 gm is placed on the horizontal plane and joined one end of a string which passes through a smooth pulley fixed at the common edge of the two planes. A mass of 100 gm is joind to the other end of the string and is placed on the incilined plane. If both branches of the string are perpendicular to the common edge find the acceleration of the system and the tenstion in the string given that the coefficients of frication between the frist body and the horizonal plane is $\frac{1}{4}$ and between the body and the incilined second plane is $\frac{1}{2\sqrt{3}}$. If the string is cut after 4 seconds from the start of motion find the total distance that the 60 gm. mass moves before coming to rest.

[$\frac{245}{4}$ cm/sec2 , $\frac{75}{4}$ gm.wt, $612\frac{1}{2}$ cm.]

(5) a) Two spheres move along the horizontal st. line in opposite direction and they collided when the velocity of the first 30 cm/sec and the velocity of the second 50 cm/sec ,so that the first sphere rebounds just after impact with velocity 10 cm/sec and the second come to rest . Find the mass of the second given that the mass of the first is 5 kg ,then find the kintic energy lost due to collision

[4 kg , $\frac{7}{10}$ joule]

b) A particle of mass 10 units moves from A (2,2) to B (5,6) in direction $\overrightarrow{AB}$ under the action of a force $\overrightarrow{F} = 2\hat{i} + 6\hat{j}$, Find the work done during the motion and if the particle start its motion from rest find the kinetic energy at B.

[30]

(6) a) A ball of mass 9 gm moves along a straight line inside medium loaded with dust, such that the dust accumulates on its surface at the rate of 1 gm/sec and the ball displacement at the end of time interval is given by $\overrightarrow{S} = (\frac{1}{3}t^3 + 3t + 2)\hat{C}$ where $\hat{C}$ is a unit vector in the direction of the motion. Find the force vector acting on the ball at any instant (t) then calculate its

magnitude at $t = 2$ seconds given that the norm of the displacement measured in cm

[51 dynes]

b) A train of mass 300 tons and the power of its engin is 450 horse ascends a plane inclined at an angle of sine $\frac{1}{240}$ to the horizontal with maximum velocity 54 km/h . Find the magnitude of the resistance to the train motion , then it moves on a horizontal road , if the resistance in each of the two roads is proportion to the magnitude of the velocity of the train .

Find the magnitude of the resistance of the horizontal road and the maximum velocity on this road.

Answers of school book Exercises

Answers of school book Exercises

First : Answers of statics

Chapter (1) : friction

Exercise (1)

(1) 87,70 kg.wt

(2) 0.5 newtons downwards the plane , the body is not about to move

(3) 1.5 newtons, the body is a bout to move (4) 10, 30 newtons (5) 220 newtons

(6) 10 $\sqrt{3}$ or 20 $\sqrt{3}$ newtons (7) $\sqrt{3}$ Kg.wt

(8) $\frac{1}{2}$, 10 newtons (9) 100, 140 newtons (10) proof

(11) **first:** 5 kg.wt act up **second:** 1 kg.wt act down (12) **first:** proof

second: $f_1 = \frac{\sqrt{3}}{2}$ w and the direction probably inclined to $\overrightarrow{f_2}$ by angle of measure 60°

Chapter (2) moments

Exercise (2 - 1)

(1) 100 cm2 , -10 cm (2) 30 cm , -40 cm (3) $\overrightarrow{F} - \frac{\overrightarrow{f} \cdot \overrightarrow{b}}{b_2} \overrightarrow{b}$

(4) $\frac{1}{2}(\overrightarrow{a} \times \overrightarrow{b})$

(6) $7\hat{k}$, 0 , 0 , $57\hat{i} - 57\hat{j}$, $45\hat{i} - 27\hat{j}$, $12\hat{i} - 30\hat{j}$ where $\hat{k}$ is a unit vector so that the vectors [$\hat{i}$, $\hat{j}$, $\hat{k}$] form right hand system

(7) $47\hat{k}$, 23.5

Exercise (2 - 2)

(1) $-5\hat{k}$, $\frac{1}{2}\sqrt{10}$ (2) $11\hat{k}$, $\frac{11}{\sqrt{3}}$

(3) $\overrightarrow{0}$, the line of action of the force passes through the origin

(4) 1 (5) $\frac{13}{9}$, $-\frac{7}{9}$ (7) zero , 100 $\sqrt{3}$

(8) $-6\hat{k}$, $14\hat{k}$, $2\hat{k}$, sum of moments $= 10\hat{k}$, $\overrightarrow{R} = 20\hat{i} + 30\hat{j}$

(9) 30 , 36 , 20 newton.meter (10) $36\sqrt{3}$, $30\sqrt{3}$ newtons.meters

(11) $35\sqrt{3}$, $75\sqrt{3}$ newtons.meters (12) BD =1.5 cm

(13) m = 24 gm.wt , $F = \frac{28}{3}$

(14) 570 kg.wt . cm

(15) they are equal and each of them = 28000 newton .cm

(16) F =15 newtons

(17) 150 newton. meter , 300 newton directed to inside of the square

(18) 126 gm.wt

Chapter (3) : parallel coplanar forces

Exercise (3 - 1)

(1) **first :** R = 110 newtons and at distance $\frac{350}{11}$ cm from A

Second : R = 30 newtons and at distance $\frac{200}{3}$ cm from A

(2) 40 newtons , 10 newtons

(3) 100 newtons and at a distance of 100 cm from the other force

400 newtons and at a distance of 25 cm from the other force

(4) F = 150 newtons , 119 cm , F = 850 newtons , 21 cm

(5) F = 40 newtons , 30 cm

(6) proof (7) proof

(8) 20 kg.wt. at a distance of 12 cm from A , in direction of the first force (9) proof

(10) R = 8 newtons acting at B and in direction parallel to the two forces 8 ,10 newtons

(11) AX = 163 cm (12) F = 19 , 8 newtons

Exercise (3 - 2)

(1) 5 newtons , 15 newtons

(2) At a distance of 80cm from each both sides

(3) 9cm , 17cm (4) 6 , 7 newtons (5) 40 , 10 newtons

(6) 2.05 , 1.95 kg.wt. At a distance 80cm from each both sides

(7) 5 , 7 kg.wt.

(8) At a distance of 24 cm from the string of greater tension

(9) 4 , 2 , 12 newtons (10) At a distance of 5cm from B , 20 ,40 newtons

(11) 6 , 4 , 15 , 25 newtons (12) 7.5 newtons at A , 62.5 newtons at B , 15 , 85 newtons

(13) 70 newtons in the string , 10 newtons on the support , 20 , 10 newtons

(14) **first :** 433 gm.wt. **second:** w =600 gm.wt.

(15) 60 cm from A , 24 gm.wt. (16) The weight of the roof = 15gm.wt.

(17) T = 950 gm.wt. , R = 100 gm.wt. T = 1283 $\frac{1}{3}$ gm.wt and w = 233 $\frac{1}{3}$ gm.wt

Chapter(4) General Equilibrium

Exercise (4 - 1)

(1) $\frac{3}{4}$ w , $\frac{5}{4}$ w , 2 w

(2) At a distance from A equals $\frac{1}{4}$ the length of the ladder

(3) Proof (4) 10.5 , 40.5 kg.wt

(5) 5 , 3 $\sqrt{2}$ newtons in the direction inclined to the horizontal by angle of measure 45°

(6) 40 , 20 $\sqrt{3}$ newtons along $\overrightarrow{AB}$

(7) 200 , 100 $\sqrt{7}$ newtons in direction inclined to the horizontal by angle whose tangent $\frac{2}{3}\sqrt{3}$

(8) $\tan\theta = \frac{3}{2}$ (9) proof (10) $\mu = \frac{\sqrt{3}}{2}$, $\frac{15\sqrt{3}}{2}$ newtons

(11) $\frac{17}{24}$ w (12) $\tan\theta = \frac{5}{12}$ (13) 60 newtons (14) Proof

Answers of school book Exercises

Chapter(5) : Couples

Exercise (5 - 1)

(1) In a the moment of the two couples is 12000, -12000 units of moment (in equilibrium).

In b the moment of the two couples is 1500, -1500 units of moment (in equilibrium).

In c the moment of the two couples is 1800, -1400 units of moment (not in equilibrium).

(2) $F \times \frac{1}{\sqrt{2}} \times 60 = 2 \times 50 \quad \therefore F = \frac{5\sqrt{2}}{3}$ kg.wt

(3) 125 newtons

(4) 2.4 kg.wt , 30° or 150° with the vertical

(5) 12 newtons

(6) 300 gm.wt , 45°

(7) 1500 newton.cm

(8) 30° with the horizontal , 90° with the horizontal

Exercise (5 - 2)

(1) 120 kg.wt.cm

(2) 45 L approximately

(3) 1040 kg.wt.cm

(4) $300\sqrt{3}$ newton.cm

(5)3 gm.wt

(6) $10\sqrt{3}$ kg.wt.cm

(7) 210 newton.cm , $\frac{7\sqrt{2}}{2}$, $\frac{7\sqrt{2}}{2}$ in direction $\overrightarrow{CA}$, $\overrightarrow{AC}$

(8) 6 , 6 newton

(9) $3\sqrt{3}$ gm.wt.cm

(10) $900\sqrt{3}$ gm.wt.cm

(11) $300\sqrt{7}$ newton.cm

(12) 648 newton.cm

(13) 84 newton.cm , 5 newton

(14) 972 newton.cm , 135 , 135 newton

Answers of school book Exercises

Second : Answers of dynamics

Chapter (1) Newton laws of motion

Exercise (1 - 1)

(1) 20 seconds (2) 25 seconds (3) 0.5×10^8 gm.m/sec (4) 4800 gm.m/sec

(5) $\simeq 7.13 \times 10^4$ gm.m/sec (6) 840 gm.m/sec (7) 21000 gm.m/sec

(8) 10 m/sec (9) 300 kg .m/sec , 316.67 kg .m/sec (10) proof

Exercise (1 - 2)

(1) d , f (2) 150 gm.wt (3) 95 gm.wt

(4) $50\sqrt{3}$ kg.wt (5) 135 kg.wt (6) $-3 \text{ m } \hat{i} \ -4 \text{ m } \hat{j}$

(7) 30 km/h (8) 75 km/h

(9) M = 400 gm.wt in the opposite direction of $\overrightarrow{F}$ i.e its line of action inclines to the two string by an angle of measure 120°

Exercise (1 - 3)

(1) $\overrightarrow{a} = \hat{i} + 5\hat{j}$, $a = \sqrt{26}$ (2) A = 0 , B = 5 (3) A = -2 , B = 1

(4) 2 metre/sec in direction making 54° with the horizontal, 1kg.wt vertically downward

(5) 2.5 m/sec², $50\sqrt{3} + 196$ newtons (6) 12.25 m/sec² (7) 4.08 m/sec approximately

(8) 104 newtons (9) 60 dynes (10) 24.5 metres

(11) 800 kg.wt (12) 402 kg.wt (13) 210 mg.wt , $\frac{1}{2}$ sec

(14) 4730 kg.wt , 0.1973 m/sec² , 18.75 minutes

(15) 70 cm/sec², 40 m , 525 m (16) a = 4 and b = 6

Exercise (1 - 4)

(1) 588 , 690 , 420 newtons (2) 2 , 2.2 , 1.8 kg.wt (3) 63 kg

(4) 1.96 m/sec² downwards

(5) Descending with acceleration of magnitude 80 cm/sec^2

(6) 14 kg , 140 cm/sec^2 , 17 kg.wt. (7) 1 kg , 215.6 cm/sec^2

(8) 250 $\sqrt{3}$ gm.wt. 490 cm/sec^2 (9) 500 $\sqrt{3}$ gm.wt , 510 cm/sec^2

(10)1.5 kg.wt , 245 $\sqrt{3}$ cm/sec^2 downward along the line of the greatest slope

(11)1 sec (12) 392 cm/ sec^2 , 522 $\frac{2}{3}$ cm (13) 5 kg.wt/ton

(14) 5 $\frac{5}{6}$ kg.wt/ton , 0.0049 m/sec^2 (15) 14 $\sqrt{2}$ m/sec

Chapter (2)

Applications of newton's law- Motion ona rough plane

Exercise (2 - 1)

(1) 4.2 m/sec^2, 5.6 newtons (2) 20 cm/sec^2 , 2.4×10^5 dynes , 20 cm

(3) 60 cm/sec (4) $\frac{98}{90}$ m/sec^2 , $\frac{49}{15}$ m/sec

(5) 4.2 m/sec^2 , Y_1 =112 newtons , Y_2 = 94.08 newtons

(6) 20 cm/sec^2 , 19200 dynes , 19200 $\sqrt{2}$ dynes

(7) 1.8 m/sec^2 , 3 m/sec

(8) $\frac{49}{75}$ m/sec^2 , 3.136 newtons , $\frac{2401}{750} \simeq$ 3.2 newtons

(9) proof (10) 2.45 m/sec^2 , 14.7 newtons , 14.7 $\sqrt{3}$ newtons

(11) 122.5 cm/sec , 22050 dynes , 22050 $\sqrt{3}$ dynes

(12) 98 cm/sec^2 , 2 sec

(13) 1.4 m/sec^2 , 25.2 $\sqrt{3}$ newtons , 180 cm (14) proof (15) 70 gm

Exercise (2 - 2)

(1) a = 0 (2) 245 cm/sec^2 , 22.05 gm .wt

(3) 140 cm/sec , 3430cm (4) 15 cm

(5) first : 4.2 $\sqrt{10}$ m/sec , second : 2.8 $\sqrt{15}$ m/sec

(6) 4.2 m/sec , $2\frac{1}{7}$ sec (7) proof (8) 245 cm/sec^2

(9) 70 cm/sec , 5 cm (10) 378 cm

(11) 70 gm , $\frac{1960}{11}$ cm/sec^2 , $\frac{3780}{11}$ gm.wt

(12) 210 cm/sec^2 the body reaches to the pulley

(13) 61.25 cm/sec^2 , 18.75 gm.wt , 612.5 cm (14) $\frac{490}{3}$ cm/sec^2

Chapter (3): Impulse and collision

Exercise (3 - 1)

(1) 294×10^8 dyne.sec , 100 ton.wt. (2) 5.6 newtons (3) $-\frac{21}{8}$ kg.m/sec

(4) 0.12 newton . sec. (5) 20 cm/sec

(6) 7m / sec in the same direction , 0.25 newton.sec

(7) 1.5 m/sec in the opposite direction , 0.25 newton . sec

(8) 2000 dyne . sec (9) 2.4 m/sec in direction of the second body (10) 11m/sec

(12) $\frac{200}{7}$ cm/sec (13) 1 m/sec in an opposite direction to that of its first velocity

(14) $71\frac{3}{7}$ kg.wt (15) 4030 dynes (16) 2250 dynes

(17) 30 sec , 96 cm/sec , 5 sec (18) 1 second (19) 177.8 newtons

Chapter (4): Work –Power - Energy

Exercise (4 - 1)

(1) -1.5 (2) -8 , zero (3) $\simeq 11 \times 10^{10}$ ergs

(4) zero , -1200 kg.wt.m (5) 80 newton .m (6) $80\sqrt{3}$ joules

(7) 49 joules (8) 12105 ergs (9) -48000 kg . wt .m

(10) 75000 ergs (11) 106875 kg.wt.m , 6562.5kg.wt.m , 6562.5kg.wt.m

(12) 0.53 joule (13) 0 , 617. 4 joules , and -586.53 joules

Answers of school book Exercises

Exercise (4 - 2)

(1) 51 (2) 13.61 kwatts = 18.52 horses (3) 250 horses

(4) 27 km/h (5) 18 km/h (6) $\simeq$ 8.18 km/h (7) 8 km/h

(8) 1500 kg.wt , 43.2 km/h (9) 240 kg.wt (10) 10 horses , $\simeq$ 4.7 km/h

(11) 1125 horses (12) 15 horses , 135 km/h (13) 50 horses

Exercise (4 - 3)

(1)11250 newton.metre (2) 10^8 ergs (3) 625000 ergs (4) 54 km/h

(5) 2250 joules (6) 2.5×10^5 ergs (7) 1200 joules (8) 300 gm

(9) 4.9 newton.metre (10) $\simeq$ 6 joules (11) 4.9 joules (12) -17.2 joules , zero

Exercise (4 - 4)

(2) 98 joules (3) $\simeq$ 2 joules (4) 5 joules , 16.76 joules (5) 4410 kg.wt

(6) 350 m/sec (7)100 m/sec (8) 6.25×10^7 ergs (9) 400 m/sec

(10) $\simeq$ 34 m/sec (11) 73.5 joules (12) 1.56 joules (13) 1960 joules

(14) 24.5 m/sec , 12.25 m/ sec (15) 1372×10^6 ergs , 20 cm/sec , $153\frac{1}{8}$ kg/m

(16) 10 cm/sec , 72000 ergs (17) 1.6 metres

(18) 7 metre/sec , 13720 joules , 36400 kg.wt

Exercise (4 - 5)

(1) 132.3 joules (2) 5.88×10^7 ergs (3) 3.5×10^5 kg.wt.m or 3.45×10^6 joules

(4) 900 kg.wt.m , 8820 joules (5) $\simeq -1.69 \times 10^7$ ergs

(6) 1.2 kg.wt.m = 1.176 joules (7) 75.50 kg.wt.m

(8) 3375 gm.wt.cm , $210\sqrt{2}$ cm/sec (9) -88 joules

(10) 13.5975 joule

المواصفات الفنية:

مقاس الكتاب:	$\frac{1}{8}$ (٥٧ × ٨٢) سم
طبع المتن:	٤ لون
طبع الغلاف:	٤ لون
ورق المتن:	٨٠ جم أبيض
ورق الغلاف:	٢٠٠ جم كوشيه
عدد الصفحات بالغلاف:	٢٧٦ صفحة

دار مكة المكرمة للطباعة والنشر